# Compactness and Noncompactness of Yamabe-type Problems on Manifolds with Boundary

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#### ABSTRACT

The study of semilinear partial differential equations has proven to be of great importance in the fields of Physics and Geometry. Solutions to such equations correspond in particular to standing waves in Schrödinger equations and metrics with constant curvature on Riemannian manifolds. In this manuscript, we obtain existence results of blowing-up solutions as well as compactness results to Yamabe-type equations on manifolds. In particular, in a joint work with Jérôme Vétois, we construct blowing-up solutions to a Yamabe-type equation on the standard sphere with dimension n = 4. Then, we construct blowing-up solutions on the standard half-sphere in dimensions  $n \ge 3$ . Finally, joint with Sérgio Almaraz and Olivaine Queiroz, we prove a compactness theorem to the boundary Yamabe problem in the scalar flat case in dimension n = 3.

## ABRÉGÉ

L'étude des équations aux dérivées partielles semi-linéaires s'est révélée d'une grande importance dans les domaines de la physique et de la géométrie. Des solutions de telles équations correspondent en particulier à des ondes stationnaires dans les équations de Schrödinger et à des métriques de courbure constante sur des variétés riemanniennes. Dans ce manuscrit, nous obtenons des résultats d'existence de solutions explosives ainsi que des résultats de compacité pour des équations de type Yamabe sur les variétés. En particulier, dans un travail en collaboration avec Jérôme Vétois, nous construisons des solutions explosives pour une équation de type Yamabe sur la sphère standard en dimension n = 4. Ensuite, nous construisons des solutions explosives sur la demi-sphère standard en dimension  $n \ge 3$ . Finalement, en collaboration avec Sérgio Almaraz et Olivaine Queiroz, nous prouvons un théorème de compacité pour le problème de Yamabe à bord dans le cas à courbure scalaire nulle en dimension n = 3.

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## PREFACE

The present manuscript contains original research work in the field of geometric analysis and partial differential equations. In particular, it studies the question of compactness and noncompactness of Yamabe-type problems on manifolds with and without boundary. The manuscript consists of an introduction and a collection of three research papers.

Chapter 1 is an introduction where we introduce the main problems, briefly review the literature and state the main results of this thesis.

Chapter 2 deals with the construction of blowing-up solutions to a cubic nonlinear Schrödinger equation on the 4-sphere. This chapter is based on the following paper (co-authors contributed equally to the paper):

 Jérôme Vétois and Shaodong Wang, Infinitely many solutions for cubic nonlinear Schrödinger equations in dimension four, Adv. Nonlinear Anal. (2017).

Chapter 3 deals with the construction of blowing-up solutions to a Yamabe-type problem on the half-sphere for  $n \ge 3$ . This chapter is based on the following paper:

 Shaodong Wang, Infinitely many blowing-up solutions for Yamabe-type problems on manifolds with boundary, Commun. Pure Appl. Anal. 17 (2017), no. 1, 209-230.

In Chapter 4, we present a compactness result to the boundary Yamabe problem in the scalar flat case. This chapter is based on the following paper (co-authors contributed equally to the paper): • Sérgio Almaraz, Olivaine S. de Queiroz and Shaodong Wang, A compactness theorem for scalar-flat metrics on 3-manifolds with boundary, J. Funct. Anal. (2018).

Please note that except Chapter 1, each chapter is self-contained and does not refer to other chapters.

## CHAPTER 1 INTRODUCTION

The present manuscript is concerned with the compactness, i.e., the existence of a priori bounds, and noncompactness of the set of positive solutions to Yamabe-type problems on manifolds. In Section 1.1 of our introduction, we will briefly review the literature on Yamabe-type problems on manifolds. Sections 1.2 and 1.3 deal with the constructions of blowing-up solutions on the standard sphere and half-sphere, respectively. In Section 1.4, we present a compactness result to a boundary Yamabe problem in the scalar flat case.

#### 1.1 Yamabe-type problems

Let (M, g) be a smooth, compact, *n*-dimensional Riemannian manifold without boundary,  $n \geq 3$ . Denote  $\Delta_g := -\operatorname{div}_g \nabla$  the Laplace-Beltrami operator. We are interested in the following semilinear elliptic partial differential equation

$$\Delta_g u + fu = u^{2^* - 1}, \ u > 0, \ \text{in } M, \tag{1.1}$$

where  $2^* := 2n/(n-2)$  is the critical exponent of the Sobolev embedding.

Equation (1.1) plays an important role both in Physics and Differential Geometry. Solutions to equation (1.1) correspond to the standing waves of nonlinear Schrödinger equations. On the other hand, when  $f \equiv \frac{n-2}{4(n-1)}R_g$ , where  $R_g$  is the scalar curvature, equation (1.1) is the Yamabe problem on (M, g). The existence of such solutions was solved in a series of works by Yamabe [63], Trudinger [53], Aubin [7] and Schoen [47]. We refer to the book by Hebey [34] for a more extensive coverage on this type of equations.

Schoen [48] in 1988 raised the question of compactness of the set of solutions to the Yamabe problem on manifolds without boundary and obtained the first compactness results, solving in particular the case of locally conformally flat manifolds. Later, Druet [20,21], Li and Zhang [38–40], Li and Zhu [42], Marques [43] and Schoen and Zhang [51] proved compactness up to dimensions  $3 \le n \le 11$ . Brendle [11] and Brendle and Marques [13] then constructed counterexamples for  $n \ge 25$ . Finally, Khuri, Marques and Schoen [36] proved compactness for all dimensions  $3 \le n \le 24$ .

Perturbations of the Yamabe problem have been studied extensively over the years. Druet [21] obtained the compactness of the set of solutions to (1.1) (see also Li and Zhu [42] in case n = 3). On the other hand, when  $f > \frac{n-2}{4(n-1)}R_g$ , we refer to Chen, Wei and Yan [16], Druet and Hebey [22], Esposito, Pistoia and Vétois [27], Hebey and Wei [35] and Thizy and Vétois [52] for blowing-up constructions.

In 1992, Escobar proposed the boundary version of the Yamabe problem. Given a smooth, compact Riemannian manifold (M, g) of dimension  $n \ge 3$  with boundary  $\partial M$ , Escobar in [25, 26] asked the following two questions:

(1) Is there a metric  $\tilde{g}$  conformally equivalent to g such that  $(M, \tilde{g})$  has constant scalar curvature and zero mean curvature?

(2) Is there a metric  $\tilde{g}$  conformally equivalent to g such that  $(M, \tilde{g})$  has zero scalar curvature and constant mean curvature?

Denote  $h_g$  the mean curvature of (M, g) with respect to the inner normal. Then the boundary Yamabe problem is equivalent to finding a conformal metric  $\tilde{g}$  such that either  $R_{\tilde{g}} = c$  and  $h_{\tilde{g}} = 0$  or  $R_{\tilde{g}} = 0$  and  $h_{\tilde{g}} = c$  for some constant c. See Almaraz [1], Brendle and Chen [12], Chen [15], Escobar [25, 26] and Marques [44, 45] for existence results.

Unlike the case of manifolds without boundary, the question of compactness of the boundary Yamabe problem remains open in general despite several works on this topic. On this problem, we refer for instance to Almaraz [2], Disconzi and Khuri [19], Felli and Ahmedou [28, 29] and Han and Li [33] for compactness results, and Almaraz [3] and Disconzi and Khuri [19] for blowing-up constructions. We also refer to Ghimenti, Micheletti and Pistoia [30–32] who recently obtained existence results of blowing-up solutions to perturbations of the Yamabe problem on manifolds with boundary.

#### **1.2** Blowing-up constructions on the sphere

In this section, we will present our results on the construction of blowing-up solutions to a Yamabe-type equation on the standard sphere. This section is based on [54], joint work with Jérôme Vétois, and we refer to Chapter 2 for more details.

Assume in this section that (M, g) is the standard sphere  $(\mathbb{S}^n, g_0)$  of dimension *n*. We study the following equation

$$\Delta_g u + f u = u^{2^* - 1}, \ u > 0, \ \text{in } M, \tag{1.2}$$

where  $2^* := 2n/(n-2)$ .

When  $f \equiv \frac{n-2}{4(n-1)}R_g \equiv \frac{n(n-2)}{4}$ , (1.2) is the Yamabe problem on  $(\mathbb{S}^n, g_0)$ . In this case, all the solutions are classified by Obata [46] and Caffarelli, Gidas and Spruck [14].

When  $f < \frac{n-2}{4(n-1)}R_g$ , Druet [21] obtained compactness results for general manifolds without boundary. If we assume moreover that  $0 < f < \frac{n-2}{4(n-1)}R_g$  is a constant, then Bidaut-Véron and Véron [10] showed that  $f^{\frac{n-2}{4}}$  is the only positive solution to (1.2). On the other hand, when  $f > \frac{n-2}{4(n-1)}R_g$ , Druet [20] showed that compactness results hold for families of solutions  $(u_{\varepsilon})_{\varepsilon>0}$  to (1.2) provided that the solutions have bounded energies, i.e.,  $||u_{\varepsilon}|| \leq C$  for some constant C independent of  $\varepsilon$ . However, if one removes the energy bound, blowing-up solutions can be constructed. Such solutions were constructed by Chen, Wei and Yan [16] on the standard sphere for  $n \geq 5$ . We extend their result to the case of dimension 4 by proving the following:

**Theorem 1.2.1.** Assume that (M, g) is the standard sphere, n = 4. If f > 2 is a constant, then there exists a family of positive solutions  $(u_{\varepsilon})_{\varepsilon>0}$  to (1.2) such that  $\|\nabla u_{\varepsilon}\|_{L^{2}(\mathbb{S}^{4})} \to \infty$  as  $\varepsilon \to 0$ .

In case n = 3, Li and Zhu [42] obtained a priori bounds on the energy of positive solutions to (1.1). Thus the dimension n = 4 is optimal in Theorem 1.2.1.

It is interesting to mention that our blowing-up constructions (Theorem 1.2.1) yield surprising applications to the construction of biharmonic maps on the 4-sphere. Indeed just recently, using our result, Baird and Ou [8] were able to show the existence of proper biharmonic maps, i.e., maps that are critical to the energy functional of a tension field, on  $\mathbb{S}^4$ . In their paper, they showed that this problem can be reduced to

proving the existence of positive solutions to Yamabe-type equations, which is given by our constructions in the case of the sphere.

In fact, a more general result can be proved in the Euclidean space. Before stating this result, let us introduce some definitions. We denote  $D^{1,2}(\mathbb{R}^4)$  the completion of smooth functions with compact support in  $\mathbb{R}^4$  with respect to the norm  $\|u\|_{D^{1,2}(\mathbb{R}^4)} = \|\nabla u\|_{L^2(\mathbb{R}^4)}$ . We say that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$  if

$$\int_{\mathbb{R}^4} \left( \left| \nabla u \right|^2 + f u^2 \right) dx \ge C \left\| u \right\|_{D^{1,2}(\mathbb{R}^4)}^2 \qquad \forall u \in D^{1,2}\left( \mathbb{R}^4 \right)$$

for some constant C > 0. The standard bubbles are functions of the form

$$U_{x,\mu}(y) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{\mu}{1+\mu^2|y-x|^2}\right)^{\frac{n-2}{2}}$$
(1.3)

for  $\mu > 0$  and  $x, y \in \mathbb{R}^n$ . It is well known that they are all the solutions to the problem

$$\Delta u = u^{\frac{n+2}{n-2}}, \ u > 0, \ \text{in } \mathbb{R}^n, \tag{1.4}$$

where  $\Delta$  is the Euclidean Laplacian with negative sign. Our general result in the Euclidean space is the following:

**Theorem 1.2.2** (Jerome-W, 17). Assume that  $(M, g) = (\mathbb{R}^4, \delta_0)$  and  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  is radially symmetric about the point 0 where  $\delta_0$  is the standard Euclidean metric and  $0 < \alpha < 1$ . Assume moreover that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$  and the function  $r \mapsto r^2 f(r)$  has a strict local maximum point  $r_0 > 0$  such that  $f(r_0) > 0$ . Then there exists a family of positive solutions  $(u_{\varepsilon})_{\varepsilon>0}$  in  $C^{2,\alpha}(\mathbb{R}^4) \cap D^{1,2}(\mathbb{R}^4)$  of (1.1) such that  $\|\nabla u_{\varepsilon}\|_{L^2(\mathbb{R}^4)} \to \infty$  as  $\varepsilon \to 0$ .

The proof of Theorem 1.2.2 relies on a Lyapunov-Schmidt-type method and a symmetric construction of infinitely many bubbles inspired by Chen, Wei and Yan [16]. We construct blowing-up solutions as the sum of standard bubbles located around a circle plus an error term. The parameters used in our construction are the location and number of bubbles. In our work, unlike the case when  $n \ge 5$ , we find that the number of bubbles behaves as a logarithm instead of a power of the bubble's height. Refined estimates are obtained in order to reduce our problem to finding a critical point of the energy functional, which then gives the existence results.

Once we prove Theorem 1.2.2, Theorem 1.2.1 follows as a corollary. The proof of Theorem 1.2.1 and Theorem 1.2.2 will be given in Chapter 2.

#### 1.3 Blowing-up constructions on the half-sphere

To study compactness properties of the Yamabe-type problem on manifolds with boundary, it is natural to first start looking at the standard half-sphere. In this section, we discuss a blowing-up construction to a perturbation problem on the half-sphere. This section is based on [55] and we refer to Chapter 3 for more details.

We turn our focus to the following problem

$$\begin{cases} \Delta_g u + fu = 0 & \text{in } M\\ \frac{\partial u}{\partial \nu} + hu = u^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases}$$
(1.5)

where (M, g) is a smooth, compact, *n*-dimensional Riemannian manifold with boundary,  $n \geq 3$ , and  $\nu$  is the outward pointing normal vector.

When  $f \equiv \frac{n-2}{4(n-1)}R_g$  and  $h \equiv \frac{n-2}{2}h_g$ , where  $R_g$  and  $h_g$  are the scalar and mean curvature of M and  $\partial M$ , respectively, problem (1.5) is the Yamabe problem with

prescribed scalar curvature 0 and mean curvature  $\frac{2}{n-2}$ . Almaraz in [2] proved a compactness result to this problem when  $n \ge 7$  under the condition that the tracefree second fundamental form is nonzero everywhere. Under the same conditions, Ghimenti, Micheletti and Pistoia [30,31] considered perturbation problems of (1.5). They obtained existence results of positive blowing-up solutions to (1.5) with perturbations on the nonlinearity in [30] and on the potential in [31].

In general dimensions  $n \ge 3$ , our main result to (1.5) on the standard half-sphere is the following:

**Theorem 1.3.1.** Let (M, g) be the standard half-sphere of dimensions  $n \ge 3$ . If  $f \equiv \frac{n-2}{4(n-1)}R_g$  and h is a positive constant, then there exists a family of positive solutions  $(u_{\epsilon})_{\epsilon>0}$  to (1.5) with  $\|\nabla u_{\epsilon}\|_{L^2(M)} \to \infty$  as  $\epsilon \to 0$ .

We obtain a more general result in the Euclidean half-space. We introduce some notations first. We denote  $D^{1,2}(\mathbb{R}^n_+)$  the completion of smooth functions with compact support in  $\mathbb{R}^n_+$  with respect to the norm  $||u||_{D^{1,2}(\mathbb{R}^n_+)} = ||\nabla u||_{L^2(\mathbb{R}^n_+)}$ . We say that  $\frac{\partial}{\partial \mu} + h$  is coercive in  $D^{1,2}(\mathbb{R}^n_+)$  if

$$\int_{\mathbb{R}^{n}_{+}} |\nabla u|^{2} + \int_{\partial \mathbb{R}^{n}_{+}} hu^{2} \ge C ||u||_{D^{1,2}(\mathbb{R}^{n}_{+})}^{2}$$

for some C > 0. The standard bubbles in this case are functions of the form

$$U_{x,\mu}(y) = B\left(\frac{1}{\mu(y_n + \frac{1}{\mu})^2 + \mu|\bar{y} - \bar{x}|^2}\right)^{\frac{n-2}{2}}$$
(1.6)

for  $\mu > 0$  where  $B = (n-2)^{\frac{n-2}{2}}$ ,  $x = (\bar{x}, x_n)$ ,  $y = (\bar{y}, y_n) \in \mathbb{R}^n_+$ . It is proved in Li and Zhu [41] that (1.6) are all the positive solutions to the following problem

$$\begin{cases} \Delta u = 0 & \text{ in } \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} = u^{\frac{n}{n-2}} & \text{ on } \partial \mathbb{R}^n_+. \end{cases}$$
(1.7)

Our result in the Euclidean half-space is the following:

**Theorem 1.3.2.** Let  $M = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, +\infty)$  be the Euclidean half-space of dimension  $n \geq 3$ . Assume that  $f \equiv 0$ ,  $h \in C^1(\partial \mathbb{R}^n_+) \cap L^{n-1}(\partial \mathbb{R}^n_+)$  is a radial function and the operator  $\Delta$  with boundary condition  $\frac{\partial}{\partial \mu} + h$  is coercive in  $D^{1,2}(\mathbb{R}^n_+)$ . Assume moreover that rh(r) has a local strict maximum or minimum point when  $n \geq 4$  and a strict local maximum point when n = 3 at some  $r_0 > 0$  with  $h(r_0) > 0$ . Then there exists infinitely many positive solutions in  $D^{1,2}(\mathbb{R}^n_+)$  to (1.5) whose energy can be made arbitrarily large.

Recall that in the sphere case, dimension 4 is critical. While we have blowing-up constructions for all  $n \ge 4$ , such solutions do not exist in dimension 3 due to the energy bound. Quite interestingly, we notice a shift of dimension for manifolds with boundary. In case of the half-sphere, blowing-up constructions do exist for dimension n = 3. Moreover, in this case, the number of bubbles behaves as a logarithm of the bubble's height, just like the case n = 4 for the sphere. We prove Theorem 1.3.2 by using a Lyapunov-Schmidt-type argument. Suitable change of variables is used for the optimal dimension n = 3. The proof of Theorem 1.3.1 and Theorem 1.3.2 will be given in Chapter 3.

#### 1.4 A compactness result

So far we discussed some constructions of blowing-up solutions to Yamabe-type equations. In this section, we will present a compactness result which we obtain in the case of manifolds with boundary. This section is based on [5], joint work with Sérgio Almaraz and Olivaine Queiroz, and we refer to Chapter 4 for more details.

Let (M, g) be a smooth, *n*-dimensional, compact Riemannian manifold with boundary,  $n \ge 3$ . Let  $\tilde{g} = u^{\frac{4}{n-2}}g$  be the conformal metric for some u > 0. The scalar and mean curvatures of  $(M, \tilde{g})$  are calculated as

$$R_{\tilde{g}} = u^{-\frac{n+2}{n-2}} \left( R_g u + \frac{4(n-1)}{n-2} \Delta_g u \right)$$
(1.8)

and

$$h_{\tilde{g}} = u^{-\frac{n}{n-2}} \left(\frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u\right), \tag{1.9}$$

where  $\Delta_g$  is the Laplacian-Beltrami operator and  $\nu$  is the outward pointing normal vector. Thus the boundary Yamabe problem is equivalent to finding a positive solution to

$$\begin{cases} \Delta_g u + \frac{n-2}{4(n-1)} R_g u = c_1 u^{\frac{n+2}{n-2}} & \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = c_2 u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}$$
(1.10)

for some constants  $c_1$  and  $c_2$ .

Han and Li [33] obtained compactness results of (1.10) with  $c_1 > 0$  and general  $c_2$ , under the assumption that (M, g) is locally conformally flat with umbilic boundary. Later, Felli and Ahmedou [28] showed that compactness holds when  $c_1 = 0$  and  $c_2 > 0$  for locally conformally flat manifolds with umbilic boundary. When  $c_1 = 0$ , Almaraz [2] obtained compactness results under the generic condition that the tracefree part of the second fundamental form does not vanish anywhere for  $n \ge 7$ . Most recently, Disconzi and Khuri [19] were able to obtain compactness results when  $c_1 > 0$ and  $c_2 = 0$  using a similar argument to Khuri, Marques and Schoen [36]. They showed that compactness holds for  $3 \le n \le 24$  if (M, g) is not conformally equivalent to the standard half-sphere with counterexamples in case  $n \ge 25$ . However, their results still rely on the assumption of umbilicity of the boundary  $\partial M$ .

For the rest of this section, we are interested in the scalar flat case, that is,  $c_1 = 0$ . We study the following equation

$$\begin{cases} \Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{in } M\\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = K u^p & \text{on } \partial M, \end{cases}$$
(1.11)

where 1 and K is a constant which has the same sign as the Yamabe invariant that we define in the paragraph below.

The Yamabe invariant of (1.11) is defined as

$$Q(M,\partial M) := \inf \left\{ Q(u); \ u \in C^1(\bar{M}), u \neq 0 \text{ on } \partial M \right\},\$$

where

$$Q(u) := \frac{\int_M |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 dv_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\left(\int_{\partial M} |u|^{p+1} d\sigma_g\right)^{\frac{2}{p+1}}},$$

where  $dv_g$  and  $d\sigma_g$  denote the volume forms of M and  $\partial M$ , respectively.

When  $Q(M, \partial M) < 0$ , (1.11) has a unique solution. If  $Q(M, \partial M) = 0$ , (1.11) becomes linear and solutions are unique up to a scalar multiplication. Therefore the

only interesting case here for our problem of compactness is when  $Q(M, \partial M) > 0$ . Our main result in this case is the following:

**Theorem 1.4.1.** Let (M, g) be a Riemannian 3-manifold with boundary  $\partial M$ . Suppose that  $Q(M, \partial M) > 0$  and M is not conformally equivalent to the unit ball. Then, given a small  $\gamma_0 > 0$ , there exists  $C(M, g, \gamma_0) > 0$  such that for any  $p \in [1 + \gamma_0, \frac{n}{n-2}]$  and any solution u > 0 of (1.11) we have

$$C^{-1} \le u \le C$$
 and  $||u||_{C^{2,\alpha}(M)} \le C$ ,

for some  $0 < \alpha < 1$ .

The proof of Theorem 1.4.1 relies on a local sign restriction given by a Pohozaevtype identity and a boundary version of the Positive Mass Theorem. Near a blow-up point, we first approximate our solution by a standard bubble plus a correction term. The correction term is defined as a solution to a non-homogeneous linear equation. Using the estimates in the previous step, we prove a local sign condition using the Pohozaev identity. This sign condition allows us to reduce the discussion to the case of isolated simple blow-up points. Finally, using the Positive Mass Theorem obtained in Almaraz, Barbosa and de Lima [4] and the local sign restriction, the compactness result follows by a standard argument.

The Positive Mass Theorem plays an important role in the proof of compactness of Yamabe-type problems. A key ingredient used in the above-mentioned results dealing with the umbilic case is a doubling manifold trick introduced by Escobar [26]. Using this trick, Escobar was able to prove a Positive Mass Theorem for manifolds with boundary provided the boundary is umbilic. The assumption of umbilicity of the boundary allows to reduce the argument to the case of manifolds without boundary, where the standard Positive Mass Theorem can be applied.

Unlike the previous works, we use a boundary version of the Positive Mass Theorem proven by Almaraz, Barbosa and de Lima [4] for general manifolds with boundary when  $3 \le n \le 7$  as well as for spin manifolds when  $n \ge 3$ . Due to the lack of explicit expression of the Green's function in the case of manifolds with boundary, we have to find relations between the local sign restriction and the Positive Mass Theorem using a flux integral method inspired by Brendle and Chen [12]. The proof of Theorem 1.4.1 is given in Chapter 4.

## CHAPTER 2 BLOWING-UP CONSTRUCTIONS ON THE SPHERE

In this chapter, we obtain an existence result of blowing-up solutions to a cubic nonlinear Schrödinger equation on the standard sphere. This chapter is based on the following paper:

 Jérôme Vétois and Shaodong Wang, Infinitely many solutions for cubic nonlinear Schrödinger equations in dimension four, Adv. Nonlinear Anal. (2017).

#### 2.1 Introduction and main results

In this note, we consider the cubic nonlinear Schrödinger equation

$$\Delta_q u + f u = u^3 \qquad \text{in } M \tag{2.1}$$

where (M, g) is a Riemannian manifold of dimension 4,  $\Delta_g := -\operatorname{div}_g \nabla$  is the Laplace-Beltrami operator, and  $f \in C^{0,\alpha}(M), \alpha \in (0, 1)$ .

In case  $(M, g) = (\mathbb{S}^4, g_0)$  where  $g_0$  is the standard metric on the sphere  $\mathbb{S}^4$ , we obtain the following result:

**Theorem 2.1.1.** Assume that  $(M, g) = (\mathbb{S}^4, g_0)$  and f > 2 is constant. Then there exists a family of positive solutions  $(u_{\varepsilon})_{\varepsilon>0}$  to (2.1) such that  $\|\nabla u_{\varepsilon}\|_{L^2(\mathbb{S}^4)} \to \infty$  as  $\varepsilon \to 0$ .

Theorem 2.1.1 extends a result obtained by Chen, Wei, and Yan [16] in dimensions  $n \ge 5$  for positive solutions of the equation

$$\Delta_q u + f u = u^{2^* - 1} \qquad \text{in } M \tag{2.2}$$

where  $2^* := 2n/(n-2)$ . The dimension four is optimal for this result since Li and Zhu [42] obtained the existence of a priori bounds on the energy of positive solutions to (2.2) in dimension three.

It is also interesting to mention that in case  $n \notin \{3, 6\}$  and  $f > \frac{n(n-2)}{4}$  on  $\mathbb{S}^n$ (or more generally  $f > \frac{n-2}{4(n-1)}$  Scal<sub>g</sub> on a general closed manifold where Scal<sub>g</sub> is the scalar curvature), Druet [20] obtained a compactness result for families of positive soutions  $(u_{\varepsilon})_{\varepsilon>0}$  of (2.2) with bounded energies, i.e. such that  $\|\nabla u_{\varepsilon}\|_{L^2(M)} < C$  for some constant C independent of  $\varepsilon$ . The above Theorem 2.1.1 together with the result of Chen, Wei, and Yan [16] in dimensions  $n \geq 5$  show that the energy assumption in Druet's result is necessary at least in the case of the standard sphere.

In case  $f \equiv \frac{n(n-2)}{4}$  and  $(M,g) = (\mathbb{S}^n, g_0)$ , the positive solutions of (2.2) have been classified by Obata [46] (see also Caffarelli, Gidas, and Spruck [14]). In this case, the solutions are not bounded in  $L^{\infty}(\mathbb{S}^n)$  but they all have the same energy. We refer to Brendle [11], Brendle and Marques [13], Khuri, Marques, and Schoen [36] and the references therein for results on the set of solutions of (2.2) in case  $f \equiv \frac{n-2}{4(n-1)}$  Scal<sub>g</sub> and  $(M,g) \neq (\mathbb{S}^n, g_0)$ . On the other hand, in case  $f < \frac{n-2}{4(n-1)}$  Scal<sub>g</sub> on a general closed manifold, Druet [21] obtained pointwise a priori bounds on the set of positive solutions of (2.2). Remark that if moreover  $0 < f < \frac{n-2}{4(n-1)}$  Scal<sub>g</sub> is constant, then Bidaut-Véron and Véron [10] obtained that  $u \equiv f^{(n-2)/4}$  is the unique positive solution of (2.2). We refer to the books of Druet, Hebey, and Robert [23] and Hebey [34] for more results on equations of type (2.1) on a closed manifold. As in the paper of Chen, Wei, and Yan [16], we obtain Theorem 2.1.1 by proving a more general result in case  $(M, g) = (\mathbb{R}^4, \delta_0)$  where  $\delta_0$  is the Euclidean metric on  $\mathbb{R}^4$ . We let  $D^{1,2}(\mathbb{R}^4)$  be the completion of the set of smooth functions with compact support in  $\mathbb{R}^4$  with respect to the norm  $||u||_{D^{1,2}(\mathbb{R}^4)} = ||\nabla u||_{L^2(\mathbb{R}^4)}$ . For simplicity, we will denote  $\Delta := \Delta_{\delta_0}, \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\delta_0}$ , and  $|\cdot| := |\cdot|_{\delta_0}$ . We say that the operator  $\Delta + f$ is coercive in  $D^{1,2}(\mathbb{R}^4)$  if

$$\int_{\mathbb{R}^4} \left( \left| \nabla u \right|^2 + f u^2 \right) dx \ge C \left\| u \right\|_{D^{1,2}(\mathbb{R}^4)}^2 \qquad \forall u \in D^{1,2}\left( \mathbb{R}^4 \right)$$

for some constant C > 0. We obtain the following result:

**Theorem 2.1.2.** Assume that  $(M,g) = (\mathbb{R}^4, \delta_0)$  and  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  is radially symmetric about the point 0. Assume moreover that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$  and the function  $r \mapsto r^2 f(r)$  has a strict local maximum point  $r_0 > 0$  such that  $f(r_0) > 0$ . Then there exists a family of positive solutions  $(u_{\varepsilon})_{\varepsilon>0}$ in  $C^{2,\alpha}(\mathbb{R}^4) \cap D^{1,2}(\mathbb{R}^4)$  of (2.1) such that  $\|\nabla u_{\varepsilon}\|_{L^2(\mathbb{R}^4)} \to \infty$  as  $\varepsilon \to 0$ .

The proof of Theorem 2.1.2 relies on a Lyapunov–Schmidt-type method as in the paper of Chen, Wei, and Yan [16]. This method for constructing solutions with infinitely many peaks was invented and successfully used in previous works by Wang, Wei and Yan [56,57] and Wei and Yan [58–61]. A specificity in our case is that the number of peaks in the construction behaves as a logarithm of the peak's height while it behaves as a power of the peak's height in the higher dimensional case (see the paper of Chen, Wei, and Yan [16]). Due to this logarithm behavior, we need to introduce some suitable changes of variables in order to find the critical points of the reduced energy in this case (see the proof of Theorem 2.1.2 at the end of Section 2.2).

## 2.2 Proof of Theorems 2.1.1 and 2.1.2

This section is devoted to the proof of Theorems 2.1.1 and 2.1.2. For any integer  $k \ge 1$ , we let  $H_k$  be the set of all functions  $u \in D^{1,2}(\mathbb{R}^4)$  such that u is even in  $x_2, x_3, x_4$  and

$$u(r\cos(\theta), r\sin(\theta), x_3, x_4)$$
  
=  $u(r\cos(\theta + 2\pi/k), r\sin(\theta + 2\pi/k), x_3, x_4)$ 

for all r > 0 and  $\theta, x_3, x_4 \in \mathbb{R}$ . Assuming that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$ , we can equip  $H_k$  with the inner product

$$\langle u, v \rangle_{H_k} := \int_{\mathbb{R}^4} \left( \langle \nabla u, \nabla v \rangle + fuv \right) dx \qquad \forall u, v \in H_k$$

and the norm

$$||u||_{H_k} := \sqrt{\langle u, u \rangle_{H_k}} \qquad \forall u \in H_k.$$

For any  $k \geq 1$  and  $r, \mu > 0$ , we define

$$W_{k,r,\mu} := \sum_{i=1}^{k} U_{i,k,r,\mu}$$

where

$$U_{i,k,r,\mu}(x) := \frac{2\sqrt{2}\mu}{1 + \mu^2 |x - x_{i,k,r}|^2} \qquad \forall x \in \mathbb{R}^4$$

and

$$x_{i,k,r} := (r \cos(2(i-1)\pi/k), r \sin(2(i-1)\pi/k), 0, 0).$$

Moreover, we define

$$P_{k,r,\mu} := \left\{ \phi \in H_k : \sum_{i=1}^k \langle \phi, Z_{i,j,k,r,\mu} \rangle_{H_k} = 0 \quad \forall j \in \{1,2\} \right\}$$

where

$$Z_{i,1,k,r,\mu} := \frac{1}{\mu} \frac{d}{dr} [U_{i,k,r,\mu}]$$
 and  $Z_{i,2,k,r,\mu} := \mu \frac{d}{d\mu} [U_{i,k,r,\mu}].$ 

First, in Proposition 2.2.1 below, we solve the equation

$$Q_{k,r,\mu}\left(W_{k,r,\mu} + \phi - (\Delta + f)^{-1}\left((W_{k,r,\mu} + \phi)^{3}_{+}\right)\right) = 0$$
(2.3)

where  $\phi \in P_{k,r,\mu}$  is the unknown function,  $Q_{k,r,\mu}$  is the orthogonal projection of  $H_k$ onto  $P_{k,r,\mu}$ , and  $u_+ := \max(u, 0)$  for all  $u : \mathbb{R}^4 \to \mathbb{R}$ .

We will prove the following result in Section 2.3:

**Proposition 2.2.1.** Let  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  be a radially symmetric function about the point 0 and such that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$ . Then for any a, b, c, d > 0 such that a < b and c < d, there exist constants  $k_0 > 0$  and  $C_0 > 0$  such that for any  $k \ge k_0$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ , there exists a unique solution  $\phi_{k,r,\mu} \in P_{k,r,\mu}$  of (2.3) such that

$$\|\phi_{k,r,\mu}\|_{H_k} \le C_0 k/\mu.$$
 (2.4)

Moreover, the map  $(r, \mu) \mapsto \phi_{k,r,\mu}$  is continuously differentiable and if there exists a critical point  $(r_k, \mu_k) \in [a, b] \times [e^{ck^2}, e^{dk^2}]$  of the function

$$(r,\mu) \longmapsto \mathcal{I}_k(r,\mu) := I(W_{k,r,\mu} + \phi_{k,r,\mu})$$

where

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^4} \left( |\nabla u| + fu^2 \right) dx - \frac{1}{4} \int_{\mathbb{R}^4} u_+^4 dx,$$

then the function  $W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}$  is a positive solution in  $C^{2,\alpha}(\mathbb{R}^4) \cap H_k$  of the equation

$$\Delta u + fu = u^3 \qquad in \ \mathbb{R}^4. \tag{2.5}$$

Then we will prove the following result in Section 2.4:

**Proposition 2.2.2.** Let  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  be a radially symmetric function about the point 0 and such that the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$ . Then there exist constants  $c_0, c_1, c_2 > 0$  such that for any a, b, c, d > 0 such that a < b and c < d,

$$I\left(W_{k,r,\mu} + \phi_{k,r,\mu}\right) = c_0 k + c_1 f\left(r\right) \frac{k \ln \mu}{\mu^2} - \frac{c_2 k^3}{r^2 \mu^2} + o\left(\frac{k^3}{\mu^2}\right)$$
(2.6)

as  $k \to \infty$  uniformly in  $r \in [a, b]$  and  $\mu \in [e^{ck^2}, e^{dk^2}]$  where  $\phi_{k,r,\mu}$  is as in Proposition 2.2.1.

Now, we prove Theorem 2.1.2 by using Propositions 2.2.1 and 2.2.2.

Proof of Theorem 2.1.2. Since  $f(r_0) > 0$  and  $r_0$  is a strict local maximum point of the function  $r \mapsto r^2 f(r)$ , we obtain that there exists  $\delta_0 > 0$  such that

$$0 < r^{2} f(r) < r_{0}^{2} f(r_{0}) \qquad \forall r \in [r_{0} - \delta_{0}, r_{0} + \delta_{0}].$$
(2.7)

For any  $k \ge 1$  and s > 0, we define  $\mu_k(s) := e^{sk^2}$ . By applying Proposition 2.2.2, we obtain

$$\mathcal{I}_{k}(r,\mu_{k}(s)) = c_{0}k + k^{3}e^{-2sk^{2}}\left(c_{1}f(r)s - \frac{c_{2}}{r^{2}} + o(1)\right)$$
(2.8)

as  $k \to \infty$  uniformly in (r, s) in compact subsets of  $(0, \infty)^2$ . Remark that the function

$$s \longmapsto e^{-2sk^2} \left( c_1 f\left(r\right) s - \frac{c_2}{r^2} \right)$$

attains its maximal value at the point

$$s_k(r) := \frac{c_2}{c_1 f(r) r^2} + \frac{1}{2k^2}$$

for all  $k \ge 1$  and  $r \in [r_0 - \delta_0, r_0 + \delta_0]$ . We define

$$\mathcal{J}_{k}(r,t) := \mathcal{I}_{k}(r,\mu_{k}(s_{k}(r)+t)).$$

By using (2.7), we obtain that there exists  $t_0 > 0$  such that

$$t_0 < \min\left(\frac{s_k(r_0)}{2}, \frac{2}{3}\left(s_k(r_0 + \delta_0) - s_k(r_0)\right), \frac{2}{3}\left(s_k(r_0 - \delta_0) - s_k(r_0)\right)\right)$$
(2.9)

for all  $k \geq 1$ . Since  $t_0 < s_k(r_0)/2$ , it follows from (2.8) that

$$\mathcal{J}_{k}(r,t) = c_{0}k + k^{3}e^{-2(s_{k}(r)+t)k^{2}}\left(c_{1}f(r)t + o(1)\right)$$
(2.10)

as  $k \to \infty$  uniformly in  $(r, t) \in [r_0 - \delta_0, r_0 + \delta_0] \times [-t_0, t_0]$ . Since  $s_k(r) > s_k(r_0)$  and f(r) > 0, it follows from (2.10) that

$$\mathcal{J}_k\left(r, t_0\right) < \mathcal{J}_k\left(r_0, t_0/2\right) \tag{2.11}$$

and

$$\mathcal{J}_k(r, -t_0) < \mathcal{J}_k(r_0, t_0/2) \tag{2.12}$$

as  $k \to \infty$  uniformly in  $r \in [r_0 - \delta_0, r_0 + \delta_0]$ . Moreover, by using (2.9) and (2.10), we obtain

$$\mathcal{J}_k\left(r_0 \pm \delta_0, t\right) < \mathcal{J}_k\left(r_0, t_0/2\right) \tag{2.13}$$

as  $k \to \infty$  uniformly in  $t \in [-t_0, t_0]$ . It follows from (2.11)–(2.13) that the function  $\mathcal{J}_k$  has a local maximum point  $(r_k, t_k) \in [r_0 - \delta_0, r_0 + \delta_0] \times [-t_0, t_0]$  for large k. We then obtain  $\nabla \mathcal{I}_k(r_k, \mu_k(s_k(r_k) + t_k)) = 0$  and so by applying the second part of Proposition 2.2.1, we obtain that the function  $W_{k,r_k,\mu_k(s_k(r_k)+t_k)} + \phi_{k,r_k,\mu_k(s_k(r_k)+t_k)}$  is a positive solution of the equation (2.5). Moreover, by using (2.4) together with the definition of  $W_{k,r_k,\mu_k(s_k(r_k)+t_k)}$ , we easily obtain

$$\left\|\nabla\left(W_{k,r_k,\mu_k(s_k(r_k)+t_k)}+\phi_{k,r_k,\mu_k(s_k(r_k)+t_k)}\right)\right\|_{L^2}\to\infty$$

as  $k \to \infty$ . This ends the proof of Theorem 2.1.2.

Finally, we prove Theorem 2.1.1 by using Theorem 2.1.2.

Proof of Theorem 2.1.1. By using a stereographic projection, we can see that the equation (2.1) on  $(M,g) = (\mathbb{S}^4, g_0)$  is equivalent to the problem

$$\begin{cases} \Delta u + \frac{4(f-2)}{\left(1+|y|^2\right)^2} u = u^3 & \text{in } \mathbb{R}^4 \\ u \in D^{1,2}(\mathbb{R}^4). \end{cases}$$
(2.14)

It is easy to check that if f > 2 is a constant, then the potential function in (2.14) satisfies the assumptions of Theorem 2.1.2. With this remark, Theorem 2.1.1 becomes a direct corollary of Theorem 2.1.2.

#### 2.3 Proof of Proposition 2.2.1

We prove Proposition 2.2.1 in this section. Throughout this section, we assume that  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  is radially symmetric about the point 0 and the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$ .

We rewrite (2.3) as  $L_{k,r,\mu}(\phi) = Q_{k,r,\mu}(N_{k,r,\mu}(\phi) + R_{k,r,\mu})$ , where

$$L_{k,r,\mu}(\phi) := Q_{k,r,\mu} \left( \phi - (\Delta + f)^{-1} \left( 3W_{k,r,\mu}^2 \phi \right) \right),$$
  

$$N_{k,r,\mu}(\phi) := (\Delta + f)^{-1} \left( (W_{k,r,\mu} + \phi)_+^3 - W_{k,r,\mu}^3 - 3W_{k,r,\mu}^2 \phi \right),$$
  

$$R_{k,r,\mu} := (\Delta + f)^{-1} \left( W_{k,r,\mu}^3 \right) - W_{k,r,\mu}.$$

First, we obtain the following result:

**Lemma 2.3.1.** For any a, b, c, d > 0 such that a < b and c < d, there exist constants  $k_1 > 0$  and  $C_1 > 0$  such that for any  $k \ge k_1$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ ,  $L_{k,r,\mu}$  is an isomorphism from  $P_{k,r,\mu}$  to itself and

$$\left\|L_{k,r,\mu}\left(\phi\right)\right\|_{H_{k}} \ge C_{2} \left\|\phi\right\|_{H_{k}} \qquad \forall \phi \in P_{k,r,\mu}.$$

*Proof.* The proof of this result follows the same lines as in the paper of Chen, Wei, and Yan [16].  $\hfill \Box$ 

We then estimate the error term  $R_{k,r,\mu}$ . We obtain the following result:

**Lemma 2.3.2.** For any a, b, c, d > 0 such that a < b and c < d, there exist constants  $k_2 > 0$  and  $C_2 > 0$  such that

$$\|R_{k,r,\mu}\|_{H_k} \le C_2 k/\mu.$$
 (2.15)

for all  $k \ge k_2$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ .

*Proof.* For any  $\phi \in H_k$ , by integrating by parts, we obtain

$$\langle R_{k,r,\mu}, \phi \rangle_{H_k} = \int_{\mathbb{R}^4} \left( W^3_{k,r,\mu} - \Delta W_{k,r,\mu} - f W_{k,r,\mu} \right) \phi dx$$
  
=  $\int_{\mathbb{R}^4} \left( W^3_{k,r,\mu} - \sum_{i=1}^k U^3_{i,k,r,\mu} - f W_{k,r,\mu} \right) \phi dx$   
=  $O\left( \sum_{i=1}^k \int_{\mathbb{R}^4} \left( \sum_{j \neq i} \sum_{l=1}^k U_{j,k,r,\mu} U_{l,k,r,\mu} + |f| \right) U_{i,k,r,\mu} |\phi| \, dx \right).$  (2.16)

By using Hölder's inequality and Sobolev's inequality, it follows from (2.16) that

$$\|R_{k,r,\mu}\|_{H_k} = \sum_{i=1}^k \mathcal{O}\left(k\sum_{j\neq i} \left\|U_{i,k,r,\mu}^2 U_{j,k,r,\mu}\right\|_{L^{4/3}} + \|fU_{i,k,r,\mu}\|_{L^{4/3}}\right).$$
(2.17)

We start with estimating the first term in (2.17). For any  $\alpha \in \{1, \ldots, k\}$ , we define

$$\Omega_{\alpha,k,r} := \left\{ (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \left\langle (y_1, y_2, 0, 0), x_{\alpha,k,r} \right\rangle \ge \cos\left(\pi/k\right) \right\}.$$

We then write

$$\int_{\mathbb{R}^4} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx = \sum_{\alpha=1}^k \int_{\Omega_{\alpha,k,r}} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx.$$
(2.18)

We observe that if  $\alpha \neq j$ , then

$$|x - x_{j,k,r}| \ge |x - x_{\alpha,k,r}|$$
 and  $|x - x_{j,k,r}| \ge \frac{1}{2} |x_{\alpha,k,r} - x_{j,k,r}|$  (2.19)

for all  $x \in \Omega_{\alpha,k,r}$ . For any  $i, j, \alpha \in \{1, \ldots, k\}$  such that  $i \neq j$ , by using (2.18), we obtain

$$U_{i,k,r,\mu}(x)^{8/3} U_{j,k,r,\mu}(x)^{4/3} \leq \begin{cases} \frac{2^{8/3} (2\sqrt{2})^4 \mu^{4/3}}{(1+\mu^2 |x-x_{i,k,r}|^2)^{8/3} |x_{i,k,r}-x_{j,k,r}|^{8/3}} & \text{if } \alpha = i \\ \frac{2^{8/3} (2\sqrt{2})^4 \mu^{4/3}}{(1+\mu^2 |x-x_{\alpha,k,r}|^2)^{8/3} |x_{i,k,r}-x_{\alpha,k,r}|^{8/3}} & \text{if } \alpha \neq i \end{cases}$$

$$(2.20)$$

for all  $x \in \Omega_{\alpha,k,r} \setminus \{x_{\alpha,k,r}\}$ . By using (2.18) and (2.20) and straightforward estimates, we obtain

$$\int_{\mathbb{R}^{4}} U_{i,k,r,\mu}^{8/3} U_{j,k,r,\mu}^{4/3} dx = O\left(\frac{\mu^{-8/3}}{|x_{i,k,r} - x_{j,k,r}|^{8/3}} + \sum_{\alpha \neq i} \frac{\mu^{-8/3}}{|x_{i,k,r} - x_{\alpha,k,r}|^{8/3}}\right)$$
$$= O\left(\frac{\mu^{-8/3}}{|x_{i,k,r} - x_{j,k,r}|^{8/3}} + \frac{k^{8/3}}{\mu^{8/3}}\right).$$
(2.21)

It follows from (2.21) that

$$\sum_{j \neq i} \left\| U_{i,k,r,\mu}^2 U_{j,k,r,\mu} \right\|_{L^{4/3}} = \mathcal{O}\left( k \left( k/\mu \right)^2 \right)$$
(2.22)

Now, we estimate the second term in (2.18). Since  $f \in L^{\infty}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$ , by applying Hölder's inequality and straightforward estimates, we obtain

$$\int_{\mathbb{R}^{4} \setminus B(x_{i,k,r},1)} |fU_{i,k,r,\mu}|^{4/3} dx = O\left(\left(\int_{\mathbb{R}^{4} \setminus B(x_{i,k,r},1)} |U_{i,k,r,\mu}|^{4} dx\right)^{1/3}\right)$$
$$= O\left(\mu^{-4/3}\right)$$
(2.23)

and

$$\int_{B(x_{i,k,r},1)} |fU_{i,k,r,\mu}|^{4/3} dx = O\left(\int_{B(x_{i,k,r},1)} |U_{i,k,r,\mu}|^{4/3} dx\right)$$
$$= O\left(\mu^{-4/3}\right).$$
(2.24)

It follows from (2.23) and (2.24) that

$$\|fU_{i,k,r,\mu}\|_{L^{4/3}} = \mathcal{O}(1/\mu).$$
(2.25)

Finally, (2.15) follows from (2.22) and (2.25).

We can now prove Proposition 2.2.1 by using Lemmas 2.3.1 and 2.3.2.

Proof of Proposition 2.2.1. We define

$$T_{k,r,\mu}(\phi) := L_{k,r,\mu}^{-1}(Q_{k,r,\mu}(N_{k,r,\mu}(\phi) + R_{k,r,\mu})) \qquad \forall \phi \in P_{k,r,\mu}$$

and

$$V_{k,r,\mu} := \left\{ \phi \in P_{k,r,\mu} : \|\phi\|_{H_k} \le C_0 k/\mu \right\}$$

where  $C_0 > 0$  is a constant to be fixed later on. It follows from Lemmas 2.3.1 and 2.3.2 that

$$\|T_{k,r,\mu}(\phi)\|_{H_k} \le C_1 (\|N_{k,r,\mu}(\phi)\|_{H_k} + C_2 k/\mu)$$
(2.26)

for all  $k \ge k_2$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ . By integrating by parts and using Hölder's inequality, Sobolev's inequality, and straightforward estimates, we obtain

$$\langle N_{k,r,\mu} (\phi) , \psi \rangle_{H_k} = \int_{\mathbb{R}^4} \left( (W_{k,r,\mu} + \phi)^3_+ - W^3_{k,r,\mu} - 3W^2_{k,r,\mu} \phi \right) \psi dx$$
  
= O \left( \left( \|W\_{k,r,\mu} \|\_{L^4} \|\phi \|^2\_{H\_k} + \|\phi \|^3\_{H\_k} \right) \|\phi \|\_{H\_k} \right) (2.27)

for all  $\psi \in H_k$ . Proceeding as in (2.18)–(2.22), we obtain

$$\int_{\mathbb{R}^4} W_{k,r,\mu}^4 dx = O\left(\sum_{i=1}^k \int_{\mathbb{R}^4} \left( U_{i,k,r,\mu}^4 + \sum_{j \neq i} U_{i,k,r,\mu}^2 U_{j,k,r,\mu}^2 \right) dx \right)$$
  
=  $O\left(k + k \left(k/\mu\right)^4 \ln \mu\right).$  (2.28)

It follows from (2.27) and (2.28) that

$$\|N_{k,r,\mu}(\phi)\|_{H_k} = \mathcal{O}\left(k^{1/4} \|\phi\|_{H_k}^2 + \|\phi\|_{H_k}^3\right).$$
(2.29)

Letting  $C_0$  be large enough so that  $C_0 > C_1C_2$ , it follows from (2.26) and (2.29) that there exists a constant  $k_3 > 0$  such that

$$T_{k,r,\mu}\left(V_{k,r,\mu}\right) \subset V_{k,r,\mu} \tag{2.30}$$

for all  $k \ge k_3$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ . Now, we prove that if k is large enough, then  $T_{k,r,\mu}$  is a contraction map from  $V_{k,r,\mu}$  to itself, i.e.

$$\|T_{k,r,\mu}(\phi_1) - T_{k,r,\mu}(\phi_2)\|_{H_k} \le C \|\phi_1 - \phi_2\|_{H_k} \quad \forall \phi_1, \phi_2 \in V_{k,r,\mu}.$$
(2.31)

for some constant  $C \in (0, 1)$ . It follows from Lemma 2.3.1 that

$$\|T_{k,r,\mu}(\phi_1) - T_{k,r,\mu}(\phi_2)\|_{H_k} \le C_1 \|N_{k,r,\mu}(\phi_1) - N_{k,r,\mu}(\phi_2)\|_{H_k}$$
(2.32)

By integrating by parts and using Hölder's inequality, Sobolev's inequality, and (2.28), we obtain

$$\langle N_{k,r,\mu} (\phi_1) - N_{k,r,\mu} (\phi_2) , \psi \rangle_{H_k}$$

$$= \int_{\mathbb{R}^4} \left( (W_{k,r,\mu} + \phi_1)_+^3 - (W_{k,r,\mu} + \phi_2)_+^3 - 3W_{k,r,\mu}^2 (\phi_1 - \phi_2) \right) \psi dx$$

$$= O\left( \left( \|W_{k,r,\mu}\|_{L^4} + \|\phi_1\|_{H_k} + \|\phi_2\|_{H_k} \right) \right)$$

$$\times \left( \|\phi_1\|_{H_k} + \|\phi_2\|_{H_k} \right) \|\phi_1 - \phi_2\|_{H_k} \|\psi\|_{H_k} \right)$$

$$= O\left( \left( k^{1/4} + \|\phi_1\|_{H_k} + \|\phi_2\|_{H_k} \right) \right)$$

$$\times \left( \|\phi_1\|_{H_k} + \|\phi_2\|_{H_k} \right) \|\phi_1 - \phi_2\|_{H_k} \|\psi\|_{H_k} \right)$$

$$(2.33)$$

It follows from (2.33) that

$$\|N_{k,r,\mu}(\phi_1) - N_{k,r,\mu}(\phi_2)\|_{H_k} = o\left(\|\phi_1 - \phi_2\|_{H_k}\right)$$
(2.34)

as  $k \to \infty$  uniformly in  $r \in [a, b]$ ,  $\mu \in [e^{ck^2}, e^{dk^2}]$ , and  $\phi_1, \phi_2 \in V_{k,r,\mu}$ . We then obtain (2.31) by putting together (2.32) and (2.34). It follows from (2.30) and (2.31) that there exists a constant  $k_4 \ge k_3$  such that for any  $k \ge k_4$ ,  $r \in [a, b]$ , and  $\mu \in [e^{ck^2}, e^{dk^2}]$ , there exists a unique solution  $\phi_{k,r,\mu} \in V_{k,r,\mu}$  of (2.3). The continuous differentiability of  $(r, \mu) \mapsto \phi_{k,r,\mu}$  is standard.

Now, we prove the last part of Proposition 2.2.1. We let  $(r_k, \mu_k) \in [a, b] \times [e^{ck^2}, e^{dk^2}]$  be a critical point of  $\mathcal{I}_k$ . Since  $\phi_{k,r,\mu}$  is a solution of (2.3), we obtain that there exist  $c_{1,k}$  and  $c_{2,k}$  such that

$$DI(W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}) = \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \langle Z_{i,j,k,r_k,\mu_k}, \cdot \rangle_{H_k}.$$
 (2.35)

It follows from (2.35) that

$$0 = \frac{\partial \mathcal{I}_{k}}{\partial r} (r_{k}, \mu_{k})$$

$$= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{dr} [W_{k,r,\mu_{k}} + \phi_{k,r,\mu_{k}}]_{r=r_{k}} \right\rangle_{H_{k}}$$

$$= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left( \mu_{k} \sum_{\alpha=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, Z_{\alpha,1,k,r_{k},\mu_{k}} \right\rangle_{H_{k}} + \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{dr} [\phi_{k,r,\mu_{k}}]_{r=r_{k}} \right\rangle_{H_{k}} \right)$$

$$(2.36)$$

and

$$0 = \frac{\partial \mathcal{I}_{k}}{\partial \mu} (r_{k}, \mu_{k})$$

$$= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{d\mu} [W_{k,r_{k},\mu} + \phi_{k,r_{k},\mu}]_{\mu=\mu_{k}} \right\rangle_{H_{k}}$$

$$= \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} \left( \frac{1}{\mu_{k}} \sum_{\alpha=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, Z_{\alpha,2,k,r_{k},\mu_{k}} \right\rangle_{H_{k}} + \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{d\mu} [\phi_{k,r_{k},\mu}]_{\mu=\mu_{k}} \right\rangle_{H_{k}} \right).$$
(2.37)

For any  $i, \alpha \in \{1, \ldots, k\}$  and  $j, \beta \in \{1, 2\}$ , direct calculations yield

$$\langle Z_{i,j,k,r_k,\mu_k}, Z_{\alpha,\beta,k,r_k,\mu_k} \rangle_{H_k} = \Lambda_j \delta_{i\alpha} \delta_{j\beta} + o(1)$$
 (2.38)

as  $k \to \infty$  where  $\Lambda_j > 0$  is a constant and  $\delta_{i\alpha} := 1$  if  $\alpha = i$  and  $\delta_{i\alpha} := 0$  if  $\alpha \neq i$ . Moreover, since  $\phi_{k,r,\mu} \in P_{k,r,\mu}$ , we obtain

$$\sum_{i=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{dr} \left[ \phi_{k,r,\mu_{k}} \right]_{r=r_{k}} \right\rangle_{H_{k}} = -\sum_{i=1}^{k} \left\langle \frac{d}{dr} \left[ Z_{i,j,k,r,\mu_{k}} \right]_{r=r_{k}}, \phi_{k,r_{k},\mu_{k}} \right\rangle_{H_{k}}$$

and therefore by using Cauchy–Schwartz inequality and (2.4), we obtain

$$\left| \sum_{i=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{dr} \left[ \phi_{k,r,\mu_{k}} \right]_{r=r_{k}} \right\rangle_{H_{k}} \right| \\
\leq \left\| \sum_{i=1}^{k} \frac{d}{dr} \left[ Z_{i,j,k,r,\mu_{k}} \right]_{r=r_{k}} \right\|_{H_{k}} \|\phi_{k,r_{k},\mu_{k}}\|_{H_{k}} = o\left(k\mu_{k}\right). \quad (2.39)$$

Similarly, we obtain

$$\sum_{i=1}^{k} \left\langle Z_{i,j,k,r_{k},\mu_{k}}, \frac{d}{d\mu} \left[ \phi_{k,r_{k},\mu} \right]_{\mu=\mu_{k}} \right\rangle_{H_{k}} \left| \\
\leq \left\| \sum_{i=1}^{k} \frac{d}{d\mu} \left[ Z_{i,j,k,r_{k},\mu} \right]_{\mu=\mu_{k}} \right\|_{H_{k}} \|\phi_{k,r_{k},\mu_{k}}\|_{H_{k}} = o\left(k\mu_{k}^{-1}\right). \quad (2.40)$$

It follows from (2.36)–(2.40) that if k is large enough, then  $c_{1,k} = c_{2,k} = 0$ , i.e. the function  $W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}$  is a weak solution of the equation

$$\Delta u + fu = u_+^3 \qquad \text{in } \mathbb{R}^4.$$

By using the coercivity of the operator  $\Delta + f$  in  $D^{1,2}(\mathbb{R}^4)$ , we obtain that  $u \ge 0$ a.e. in  $\mathbb{R}^4$ . It then follows from standard elliptic regularity theory and the strong maximum principle that  $W_{k,r_k,\mu_k} + \phi_{k,r_k,\mu_k}$  is a strong positive solution in  $C^{2,\alpha}(\mathbb{R}^4)$ of (2.5).

## 2.4 Proof of Proposition 2.2.2

We prove Proposition 2.2.2 in this section. Throughout this section, we assume that  $f \in C^{0,\alpha}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$  is radially symmetric about the point 0 and the operator  $\Delta + f$  is coercive in  $D^{1,2}(\mathbb{R}^4)$ . First, we obtain the following result:
**Lemma 2.4.1.** There exist constants  $c_0, c_1, c_2 > 0$  such that for any a, b, c, d > 0such that a < b and c < d,

$$I(W_{k,r,\mu}) = c_0 k + c_1 f(r) \frac{k \ln \mu}{\mu^2} - \frac{c_2 k^3}{r^2 \mu^2} + o\left(\frac{k^3}{\mu^2}\right)$$
(2.41)

as  $k \to \infty$  uniformly in  $r \in [a, b]$  and  $\mu \in \left[e^{ck^2}, e^{dk^2}\right]$ .

*Proof.* By integrating by parts, we obtain

$$I\left(W_{k,r,\mu}\right) = \frac{1}{2} \int_{\mathbb{R}^{4}} \left(\Delta W_{k,r,\mu} + fW_{k,r,\mu}\right) W_{k,r,\mu} dx - \frac{1}{4} \int_{\mathbb{R}^{4}} W_{k,r,\mu}^{4} dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^{4}} \left(\sum_{i,j=1}^{k} U_{i,k,r,\mu}^{3} U_{j,k,r,\mu} + fW_{k,r,\mu}^{2} - \frac{1}{2} W_{k,r,\mu}^{4}\right) dx$$

$$= \frac{1}{2} \sum_{i=1}^{k} \int_{\mathbb{R}^{4}} \left(fU_{i,k,r,\mu}^{2} + \frac{1}{2} U_{i,k,r,\mu}^{4} - \sum_{j\neq i} U_{i,k,r,\mu}^{3} U_{j,k,r,\mu}\right) dx$$

$$+ f \sum_{j\neq i} U_{i,k,r,\mu} U_{j,k,r,\mu}\right) dx$$

$$+ O\left(\sum_{i,l=1}^{k} \sum_{j\neq i} \sum_{m\neq l} \int_{\mathbb{R}^{4}} U_{i,k,r,\mu} U_{j,k,r,\mu} U_{l,k,r,\mu} U_{m,k,r,\mu} dx\right)$$

$$= \frac{1}{2} \sum_{i=1}^{k} \int_{\mathbb{R}^{4}} \left(fU_{i,k,r,\mu}^{2} + \frac{1}{2} U_{i,k,r,\mu}^{4} - \sum_{j\neq i} U_{i,k,r,\mu}^{3} U_{j,k,r,\mu} dx\right)$$

$$+ f \sum_{j\neq i} U_{i,k,r,\mu} U_{j,k,r,\mu}\right) dx + O\left(k^{2} \sum_{i=1}^{k} \sum_{j\neq i} \int_{\mathbb{R}^{4}} U_{i,k,r,\mu}^{2} U_{j,k,r,\mu}^{2} dx\right)$$

$$(2.42)$$

Direct calculations yield

$$\int_{\mathbb{R}^4} U_{i,k,r,\mu}^4 dx = \left(2\sqrt{2}\right)^4 \int_{\mathbb{R}^4} \frac{dx}{\left(1+|x|^2\right)^4} \,. \tag{2.43}$$

and

$$\int_{\mathbb{R}^4} f U_{i,k,r,\mu}^2 dx = 16\pi^2 f(r) \,\frac{\ln\mu}{\mu^2} + o\left(\frac{\ln\mu}{\mu^2}\right)$$
(2.44)

as  $k \to \infty$  uniformly in  $r \in [a, b]$  and  $\mu \in [e^{ck^2}, e^{dk^2}]$ . By splitting the integral as in (2.18) and estimating each term, we obtain

$$\begin{split} \sum_{j \neq i} \int_{\mathbb{R}^4} U_{i,k,r,\mu}^3 U_{j,k,r,\mu} dx &= \sum_{j \neq i} \int_{\Omega_{i,k,r}} \frac{64\mu^2}{\left(1 + \mu^2 |x - x_{i,k,r}|^2\right)^3} \\ &\times \left(\frac{1 + O\left(\mathbf{1}_{\Omega_{i,k,r} \setminus B\left(x_{i,|x_{1,k,r} - x_{2,k,r}|/2}\right)\right)}{|x_{i,k,r} - x_{j,k,r}|^2} + O\left(\frac{\mu^{-2} + |x - x_{i,k,r}| |x_{i,k,r} - x_{j,k,r}|}{|x_{i,k,r} - x_{j,k,r}|^4} \mathbf{1}_{B\left(x_{i,|x_{1,k,r} - x_{2,k,r}|/2\right)}\right)\right) dx \\ &+ O\left(\sum_{\alpha \neq i} \frac{k\mu}{|x_{i,k,r} - x_{\alpha,k,r}|^3} \int_{\Omega_{\alpha,k,r}} \frac{dx}{\left(1 + \mu^2 |x - x_{\alpha,k,r}|^2\right)^{5/2}}\right) \\ &= \sum_{j \neq i} \left(\frac{64\mu^{-2}}{|x_{i,k,r} - x_{j,k,r}|^2} \int_{\mathbb{R}^4} \frac{dx}{\left(1 + |x|^2\right)^3} + O\left(\frac{k\mu^{-3}}{|x_{i,k,r} - x_{j,k,r}|^3}\right)\right) \\ &= \frac{32k^2}{\pi^2 r^2 \mu^2} \int_{\mathbb{R}^4} \frac{dx}{\left(1 + |x|^2\right)^3} \sum_{j=1}^\infty \frac{1}{j^2} + O\left(\frac{k^2}{\mu^2}\right) \end{split}$$
(2.45)

as  $k \to \infty$  uniformly in  $r \in [a, b]$  and  $\mu \in [e^{ck^2}, e^{dk^2}]$ . Moreover, straightforward estimates give

$$\begin{split} &\sum_{j \neq i} \int_{\mathbb{R}^{4} \setminus (B(x_{i,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2) \cup B(x_{j,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2))} U_{i,k,r,\mu}^{2} U_{i,k,r,\mu}^{2} dx \\ &= O\left(\mu^{-4} \sum_{j \neq i} \int_{\mathbb{R}^{4} \setminus (B(x_{i,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2) \cup B(x_{j,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2))} \right) \\ &= O\left(\sum_{j \neq i} \frac{\mu^{-4}}{|x_{i,k,r,\mu} - x_{j,k,r,\mu}|^{4}}\right) = O\left((k/\mu)^{4}\right), \end{split}$$
(2.46)

$$\sum_{j \neq i} \int_{B(x_{i,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2) \cup B(x_{j,k,r,\mu}, |x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2)} U_{i,k,r,\mu}^2 U_{j,k,r,\mu}^2 dx$$

$$= O\left(\sum_{j \neq i} \frac{\mu^{-4}}{|x_{i,k,r,\mu} - x_{j,k,r,\mu}|^4} \int_{B(0,\mu|x_{i,k,r,\mu} - x_{j,k,r,\mu}|/2)} \frac{dx}{(1+|x|^2)^2}\right)$$

$$= O\left(\sum_{j \neq i} \frac{\mu^{-4} \ln \mu}{|x_{i,k,r,\mu} - x_{j,k,r,\mu}|^4}\right) = O\left((k/\mu)^4 \ln \mu\right), \qquad (2.47)$$

$$\sum_{j \neq i} \int_{\mathbb{R}^{4} \setminus (B(x_{i,k,r,\mu},1) \cup B(x_{j,k,r,\mu},1))} fU_{i,k,r,\mu} U_{j,k,r,\mu} dx$$
  
=  $O\left(\sum_{j \neq i} \left( \int_{\mathbb{R}^{4} \setminus B(x_{j,k,r,\mu},1)} U_{j,k,r,\mu}^{4} dx \right)^{1/2} \right)$   
=  $O\left(k\left( \int_{\mathbb{R}^{4} \setminus B(0,\mu)} \frac{dx}{(1+|x|^{2})^{4}} \right)^{1/2} \right) = O\left(\frac{k}{\mu^{2}}\right),$  (2.48)

and

$$\sum_{j \neq i} \int_{B(x_{i,k,r,\mu},1) \cup B(x_{j,k,r,\mu},1)} fU_{i,k,r,\mu} U_{j,k,r,\mu} dx$$
  
=  $O\left(\sum_{j \neq i} \int_{B(x_{i,k,r,\mu},1)} U_{i,k,r,\mu} U_{j,k,r,\mu} dx\right)$   
=  $O\left(\sum_{j \neq i} \int_{B(x_{i,k,r,\mu},1)} \frac{\mu^{-2} dx}{|x - x_{i,k,r,\mu}|^2 |x - x_{j,k,r,\mu}|^2}\right)$   
=  $O\left(\mu^{-2} \sum_{j \neq i} \ln \frac{1}{|x_{i,k,r,\mu} - x_{j,k,r,\mu}|}\right) = O\left(\frac{k \ln k}{\mu^2}\right)$  (2.49)

as  $k \to \infty$  uniformly in  $r \in [a, b]$  and  $\mu \in [e^{ck^2}, e^{dk^2}]$ . Finally, (2.41) follows from (2.42)–(2.49).

We can now prove Proposition 2.2.2 by using Lemma 2.4.1.

Proof of Proposition 2.2.2. By integrating by parts, we obtain

$$I\left(W_{k,r,\mu} + \phi_{k,r,\mu}\right) = I\left(W_{k,r,\mu}\right) - \langle R_{k,r,\mu}, \phi_{k,r,\mu} \rangle_{H_{k}} + \frac{1}{2} \left\|\phi_{k,r,\mu}\right\|_{H_{k}}^{2} - \frac{1}{4} \int_{\mathbb{R}^{4}} \left(\left(W_{k,r,\mu} + \phi_{k,r,\mu}\right)_{+}^{4} - W_{k,r,\mu}^{4} - 4W_{k,r,\mu}^{3}\phi_{k,r,\mu}\right) dx. \quad (2.50)$$

By using Cauchy–Schwartz inequality, Lemma 2.3.1, and Proposition 2.2.1, we obtain

$$-\langle R_{k,r,\mu}, \phi_{k,r,\mu} \rangle_{H_k} + \frac{1}{2} \| \phi_{k,r,\mu} \|_{H_k}^2 = \mathcal{O}\left( (k/\mu)^2 \right).$$
(2.51)

Moreover, by using Hölder's inequality, Sobolev's inequality, (2.28), and Lemma 2.3.1, we obtain

$$\int_{\mathbb{R}^{4}} \left( \left( W_{k,r,\mu} + \phi_{k,r,\mu} \right)_{+}^{4} - W_{k,r,\mu}^{4} - 4W_{k,r,\mu}^{3}\phi_{k,r,\mu} \right) dx 
= O\left( \int_{\mathbb{R}^{4}} \left( W_{k,r,\mu}^{2} + \phi_{k,r,\mu}^{2} \right) \phi_{k,r,\mu}^{2} dx \right) 
= O\left( \left\| W_{k,r,\mu} \right\|_{L^{4}}^{2} \left\| \phi_{k,r,\mu} \right\|_{H_{k}}^{2} + \left\| \phi_{k,r,\mu} \right\|_{H_{k}}^{4} \right) 
= O\left( \sqrt{k} \left( k/\mu \right)^{2} + \left( k/\mu \right)^{4} \right).$$
(2.52)

Finally, (2.6) follows from (2.50)-(2.52).

# CHAPTER 3 BLOWING-UP CONSTRUCTIONS ON THE HALF SPHERE

In this chapter, we construct blowing-up solutions to a Yamabe-type problem on manifolds with boundary. This chapter is based on the following paper:

 Shaodong Wang, Infinitely many blowing-up solutions for Yamabe-type problems on manifolds with boundary, Commun. Pure Appl. Anal. 17 (2017), no. 1, 209-230.

### 3.1 Introduction and the main result

#### 3.1.1 Introduction

In this paper, we consider the nonlinear Neumann problem

$$\begin{cases} \Delta_g u + fu = 0 \ in \ M\\ \frac{\partial u}{\partial \nu} + hu = u^{\frac{n}{n-2}} \ on \ \partial M \end{cases}$$
(3.1)

where (M, g) is a Riemannian manifold with boundary,  $\Delta_g = -div_g \nabla$  is the Laplace-Beltrami operator, and  $\nu$  is the outward pointing normal vector. We are interested in the question of existence of families of positive solutions  $(u_{\varepsilon})_{\varepsilon>0}$  to problem (3.1) which blow up in the sense that  $u_{\varepsilon} \to \infty$  as  $\varepsilon \to 0$ .

In the case  $f \equiv \frac{n-2}{4(n-1)}Scal_g$  and  $h \equiv \frac{n-2}{2}h_g$  where  $Scal_g$  and  $h_g$  are the scalar and mean curvature of M and  $\partial M$  respectively, the problem (3.1) is the Yamabe problem with prescribed scalar curvature 0 and mean curvature  $\frac{2}{n-2}$ . Early references on this problem are by Escobar [25,26]. See also Almaraz [1], Brendle and Chen [12], Chen [15] and Marques [44, 45] for more recent results on the existence of solutions. In this case, compactness results, i.e., nonexistence of blowing-up solutions have been obtained by Almaraz [2], Disconzi and Khuri [19] and Felli and Ahmedou [28,29] (see also Han and Li [33] in the case where the prescribed scalar curvature is a positive constant). With regard to existence results, Almaraz [3] and Disconzi and Khuri [19] obtained exitence results of blowing-up solutions to (3.1) when  $n \ge 25$  in case f and h are the potentials of the Yamabe problem.

Ghimenti, Micheletti and Pistoia [30,31] have recently obtained existence results of positive blowing-up solutions to problem (3.1) when  $n \ge 7$  with perturbations on the nonlinearity,  $f \equiv \frac{n-2}{4(n-1)}Scal_g$  and  $h \ne \frac{n-2}{2}h_g$  (in [30]) and on the potential,  $f \equiv \frac{n-2}{4(n-1)}Scal_g$  and  $h \equiv \frac{n-2}{2}h_g$  (in [31]).

In this paper, we prove existence results of positive blowing-up solutions to (3.1) with unbounded energy (see energy functional in (3.11)) for  $n \ge 3$  when (M, g) is the standard half-sphere. Our main result is as follows:

**Theorem 3.1.1.** Let (M, g) be the standard half-sphere of dimensions  $n \ge 3$ . If  $f \equiv \frac{n-2}{4(n-1)}Scal_g$  and h is a positive constant, then there exists a family of positive blowing-up solutions  $(u_{\epsilon})_{\epsilon>0}$  to (3.1) with  $\|\nabla u_{\epsilon}\|_{L^2(M)} \to \infty$  as  $\epsilon \to 0$ .

Theorem 3.1.1 is the analogue for manifolds with boundary of the result obtained by Chen, Wei and Yan [16] for the equation

$$\Delta_g u + fu = u^{\frac{n+2}{n-2}}, u > 0$$

on the standard sphere of dimensions  $n \ge 5$  (see also Vétois and Wang [54] in case n = 4). Remark that such solutions do not exist in case n = 3 for manifolds without boundary due to the energy bound obtained by Li and Zhu [42].

There has been many works on the question of existence of blowing-up solutions of Yamabe-type problems on manifolds without boundary. In this case we refer to for instance Druet [20,21], Khuri, Marques and Schoen [36], Li and Zhang [38–40], Li and Zhu [42], Marques [43], Schoen [48,49] for compactness results, i.e., nonexistence of blowing-up solutions and Brendle [11], Brendle and Marques [13], Chen, Wei and Yan [16], Druet and Hebey [22], Esposito, Pistoia and Vétois [27], Hebey and Wei [35], Thizy and Vétois [52] for existence results. Needless to say, we do not pretend to any exhaustivity in this list. We also refer to Hebey [34] for a reference in book form on this topic.

We will prove Theorem 3.1.1 by proving the more general Theorem 3.1.2 on a half-space. The proof of Theorem 3.1.2 relies on a Lyapunov-Schmidt type method as in the paper of Chen, Wei and Yan [16]. This method for constructing solutions with infinitely many peaks was invented and successfully used in papers by Wang, Wei and Yan [56,57] and Wei and Yan [58–61]. The case when n = 3 is critical in this problem. In this case we manage to get a logarithm term in the energy estimate and the number of peaks in our construction behaves as a logarithm of the peaks' height. This is in contrast with the case when  $n \ge 4$  where the number of peaks behaves as a power of the peaks' height. We will prove Theorem 3.1.2 in Section 3.2. The reduction will be carried out in Section 3.3. In Section 3.4 we perform the estimate of the energy.

#### 3.1.2 A more general result on a half-space

We will prove Theorem 3.1.1 by proving the following more general results on a half-space. We say that the operator  $\Delta$  with boundary condition  $\frac{\partial}{\partial \mu} + h$  is coercive in  $D^{1,2}(\mathbb{R}^n_+)$  if

$$\int_{\mathbb{R}^n_+} |Du|^2 + \int_{\partial \mathbb{R}^n_+} hu^2 \ge C ||u||^2_{D^{1,2}(\mathbb{R}^n_+)}$$

for some C > 0. Here  $D^{1,2}(\mathbb{R}^n_+)$  is defined to be the completion of smooth functions with with compact support in  $\mathbb{R}^n_+$  with respect to the norm  $||u||_{D^{1,2}(\mathbb{R}^n_+)} = ||Du||_{L^2(\mathbb{R}^n_+)}$ .

**Theorem 3.1.2.** Let  $M = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, +\infty)$  be the Euclidean half-space of dimension  $n \ge 3$ . Assume that  $f \equiv 0$ ,  $h \in C^1(\partial \mathbb{R}^n_+) \cap L^{n-1}(\partial \mathbb{R}^n_+)$  is a radial function and the operator  $\Delta$  with boundary condition  $\frac{\partial}{\partial \mu} + h$  is coercive in  $D^{1,2}(\mathbb{R}^n_+)$ . Assume moreover that rh(r) has a local strict maximum or minimum point when  $n \ge 4$  and a strict local maximum point when n = 3 at some  $r_0 > 0$  with  $h(r_0) > 0$ . Then there exists infinitely many positive solutions in  $D^{1,2}(\mathbb{R}^n_+)$  to (3.1) whose energy can be made arbitrarily large.

Now we prove Theorem 3.1.1.

Proof of Theorem 3.1.1. When M is the standard half-sphere, we take the stereographic projection  $\Phi_p: \mathbb{S}^n \to \mathbb{R}^n$ :

$$\Phi_p(x) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right).$$

Now  $g = \varphi^2 g_0 = (\frac{2}{1+|x|^2})^2 g_0$  where  $g_0$  is the Euclidean metric. Let  $v = \varphi^{\frac{n-2}{2}} u$ . With a direct computation we get that problem (3.1) is equivalent to

$$\begin{cases} \Delta v = 0 \text{ in } \mathbb{R}^n_+ \\ \frac{\partial v}{\partial \nu} + \frac{2h}{1+|y|^2} v = v^{\frac{n}{n-2}} \text{ on } \partial \mathbb{R}^n_+ \end{cases}$$

It is easy to check that if h is a positive constant as in Theorem 3.1.1,  $\frac{2h}{1+|y|^2} \in C^1(\partial \mathbb{R}^n_+) \cap L^{n-1}(\partial \mathbb{R}^n_+)$  is a radial function and  $\frac{2hr}{1+r^2}$  has a strict local maximum point at  $r_0 = 1$ . Moreover, the coercivity follows from  $\frac{2h}{1+|y|^2} > 0$ . So Theorem 3.1.1 is a direct corollary of Theorem 3.1.2.

## 3.2 Proof of Theorem 3.1.2

We will prove Theorem 3.1.2 in this section. For any  $x \in \partial \mathbb{R}^n_+$  and  $\mu > 0$ , we define

$$U_{x,\mu}(y) = B\left(\frac{1}{\mu(y_n + \frac{1}{\mu})^2 + \mu|\bar{y} - \bar{x}|^2}\right)^{\frac{n-2}{2}}$$
(3.2)

where  $B = (n-2)^{\frac{n-2}{2}}$ ,  $x = (\bar{x}, x_n)$ ,  $y = (\bar{y}, y_n) \in \mathbb{R}^n_+$  and  $n \ge 3$ . It is proved in Li and Zhu [41] that (3.2) are all the positive solutions to the following problem:

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} = u^{\frac{n}{n-2}} \text{ on } \partial \mathbb{R}^n_+ \end{cases}$$

Fix k > 0 a positive integer, we define

$$H_{s} = \{ u \in D^{1,2}(\mathbb{R}^{n}_{+}) : u \text{ is even in } y_{2}, ..., y_{n-1}, u(rcos(\theta), rsin(\theta), y') \\ = u(rcos(\theta + \frac{2\pi}{k}), rsin(\theta + \frac{2\pi}{k}), y'), \theta \in \mathbb{R}, r > 0, (y_{1}, y_{2}, y') \in \mathbb{R}^{n}_{+} \}$$

and the inner product in  ${\cal H}_s$  as:

$$\langle u, v \rangle = \int_{\mathbb{R}^n_+} DuDv + \int_{\partial \mathbb{R}^n_+} huv$$

and the norm  $||u||^2 = \langle u, u \rangle$ . It follows from the continuous embedding  $D^{1,2}(\mathbb{R}^n_+) \hookrightarrow$  $L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)$  that there exists a constant  $k_n > 0$  such that

$$\left(\int_{\partial \mathbb{R}^{n}_{+}} |u|^{\frac{2(n-1)}{n-2}} dy\right)^{\frac{n-2}{n-1}} \leqslant k_{n}^{2} \int_{\mathbb{R}^{n}_{+}} |du|^{2} dy.$$
(3.3)

We let  $u = i^*(G) \in D^{1,2}(\mathbb{R}_n^+)$  be the weak solution to

$$\begin{cases} \Delta u = 0 \ in \ \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} + hu = G \ on \ \partial \mathbb{R}^n_+ \end{cases}$$

in the sense that

$$\langle \phi, i^*(G) \rangle = \int_{\partial \mathbb{R}^n_+} G\phi$$

for any  $\phi \in C_c^{\infty}(\mathbb{R}^n_+)$  and our problem (3.1) becomes:  $u = i^*(u^{\frac{n}{n-2}})$ .

For any  $i \in \{1, ..., k\}$  and r > 0, let  $x_i = (rcos \frac{2(i-1)\pi}{k}, rsin \frac{2(i-1)\pi}{k}, 0) \in \partial \mathbb{R}^n_+$ .

Define

$$Z_{i,r,\mu,1} = \frac{1}{\mu} \frac{\partial U_{x_i,\mu}}{\partial r} = B \frac{(n-2) \left\langle \bar{y} - \bar{x}_i, \frac{\partial x_i}{\partial r} \right\rangle}{\left(\mu \left((y_n + \frac{1}{\mu})^2 + |\bar{y} - \bar{x}_i|^2\right)\right)^{\frac{n}{2}}},$$
(3.4)

$$Z_{i,r,\mu,2} = \mu \frac{\partial U_{x_i,\mu}}{\partial \mu} = -\frac{B(n-2)}{2} \frac{\left(\mu(y_n + \frac{1}{\mu})^2 + \mu|\bar{y} - \bar{x}_i|^2 - 2(y_n + \frac{1}{\mu})\right)}{\left(\mu((y_n + \frac{1}{\mu})^2 + |\bar{y} - \bar{x}_i|^2)\right)^{\frac{n}{2}}},$$
(3.5)

$$W_{k,r,\mu} = \sum_{i=1}^{k} U_{x_i,\mu}(y)$$
(3.6)

and

$$P_{k,r,\mu} = \{ \Phi \in H_s : \langle \sum_{i=1}^k Z_{i,r,\mu,j}, \Phi \rangle = 0, j = 1, 2 \}.$$

We also define  $L_{k,r,\mu}$ :  $H_s \to P_{k,r,\mu}$  as a bounded linear operator by

$$L_{k,r,\mu}(u) = Q_{k,r,\mu}\left(u - i^*\left(\frac{n}{n-2}W_{k,r,\mu}^{\frac{2}{n-2}}u\right)\right)$$
(3.7)

for any  $u \in H_s$ ,  $W_{k,r,\mu}$  as in (3.6) where  $Q_{k,r,\mu}$  is the orthogonal projection map onto  $P_{k,r,\mu}$ . Throughout this paper, we let

$$\mu \in [C_1 k^{\frac{n-2}{n-3}}, C_2 k^{\frac{n-2}{n-3}}] \tag{3.8}$$

when  $n \ge 4$ ,

$$\mu \in [e^{C_3klnk}, e^{C_4klnk}] \tag{3.9}$$

when n = 3 and

$$r \in [r_0 - \delta, r_0 + \delta] \tag{3.10}$$

for some  $0 < C_1 < C_2$ ,  $0 < C_3 < C_4$  and  $\delta > 0$  to be determined later in the proof of Theorem 3.1.2. The energy functional is defined as:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} |Du|^2 + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} hu^2 - \frac{(n-2)}{2(n-1)} \int_{\partial \mathbb{R}^n_+} u_+^{\frac{2(n-1)}{n-2}}$$
(3.11)

for any  $u \in H_s$  where  $u_+ := max\{u, 0\}$ .

We will start by solving the following problem:

$$Q_{k,r,\mu}((W_{k,r,\mu} + \Phi) - i^*((W_{k,r,\mu} + \Phi)_+^{\frac{n}{n-2}})) = 0$$
(3.12)

in the projection space  $P_{k,r,\mu}$ . Then we will show that there exists a critical point for  $I(W_{k,r,\mu} + \Phi)$ . From there we will conclude the proof of Theorem 3.1.2.

We state our existence result of solutions in  $P_{k,r,\mu}$  as Proposition 3.2.1 below. We refer to Section 3.3 for the proof of this result.

**Proposition 3.2.1.** Let  $\mu, r$  be as in (3.8), (3.9) and (3.10). There exists  $k_0 > 0$ and C > 0 independent of k such that (3.12) has a unique solution  $\Phi_{k,r,\mu}$  in  $P_{k,r,\mu}$  for every  $k \ge k_0$  and

$$\|\Phi_{k,r,\mu}\| \leqslant C\frac{k}{\mu}$$

when  $n \ge 5$ ,

$$\left\|\Phi_{k,r,\mu}\right\| \leqslant C \frac{k l n \mu}{\mu}$$

when n = 4, and

$$\left\|\Phi_{k,r,\mu}\right\| \leqslant C\frac{k}{\mu^{\frac{1}{2}}}$$

when n=3. Moreover if  $(\bar{r_k}, \bar{\mu_k})$  is a critical point of  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$ , then  $W_{k,\bar{r_k},\bar{\mu_k}} + \Phi_{k,\bar{r_k},\bar{\mu_k}}$  is a positive solution to (3.1).

Once we have solved problem (3.12) in  $P_{k,r,\mu}$ , we need to perform the estimate of the energy  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$  where I is defined in (3.11). We state this estimate in Proposition 3.2.2 below. We refer to Section 3.4 for the proof of Proposition 3.2.2.

**Proposition 3.2.2.** Let  $W_{k,r,\mu}$  be defined as in (3.6), I be defined as in (3.11) and  $\Phi_{k,r,\mu} \in P_{k,r,\mu}$  be the solution we obtained from Proposition 3.2.1. Let  $\mu, r$  be as in (3.8), (3.9), (3.10). We have

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = k(a + \frac{bh(r)}{\mu} - \frac{ck^{n-2}}{\mu^{n-2}r^{n-2}} + O(\frac{1}{\mu^{1+\sigma}})),$$
  
$$\frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} = k(-\frac{bh(r)}{\mu^2} + \frac{c(n-2)k^{n-2}}{\mu^{n-1}r^{n-2}} + O(\frac{1}{\mu^{2+\sigma}}))$$

when  $n \ge 4$ , and

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = k(a + \frac{bh(r)ln\mu}{\mu} - \frac{cklnk}{\mu r} + O(\frac{k}{\mu}))$$

when n = 3 for some  $\sigma, a, b, c > 0$  independent of k.

Now during the proof of Theorem 3.1.2, we need the following Lemma 3.2.3 from Thizy and Vétois [52] to find a critical point for the energy in the presence of a saddle point when  $n \ge 4$ .

**Lemma 3.2.3** (Thizy and Vétois [52]). Let  $n_1, n_2$  be two integers,  $\Omega_1$  be a bounded and open subset of  $\mathbb{R}^{n_1}$ ,  $\Omega_2$  be a bounded open and smooth subset of  $\mathbb{R}^{n_2}$ , and  $\Omega = \Omega_1 \times \Omega_2$ . Let F be a C<sup>2</sup>-function in a neighborhood of  $\overline{\Omega}$  such that

- 1. The outward normal derivative of F on  $\Omega_1 \times \partial \Omega_2$  is positive,
- 2. There exists  $\bar{x} \in \Omega_1$  such that  $\inf_{\Omega_2} F(\bar{x}, ) > \sup_{\partial \Omega_1 \times \Omega_2} F$ .

Then F has a critical point in the interior of  $\Omega$ .

Now we prove Theorem 3.1.2.

Proof of Theorem 3.1.2. For k large we want to show that there exists a critical point  $(\bar{r}_k, \bar{\mu}_k)$  of  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$ . By Proposition 3.2.1 we know  $W_{k,\bar{r}_k,\bar{\mu}_k} + \Phi_{k,\bar{r}_k,\bar{\mu}_k}$  is a positive solution to (3.1). Here  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$  is as in Proposition 3.2.2. We first prove for  $n \ge 4$ . Let  $\mu_{0,k}(r)$  be such that

$$-\frac{bh(r)}{\mu_{0,k}^2} + \frac{c(n-2)k^{n-2}}{\mu_{0,k}^{n-1}r^{n-2}} = 0$$

namely

$$\mu_{0,k}(r) = \left(\frac{c(n-2)k^{n-2}}{bh(r)r^{n-2}}\right)^{\frac{1}{n-3}} = \left(\frac{c(n-2)}{bh(r)r^{n-2}}\right)^{\frac{1}{n-3}}k^{\frac{n-2}{n-3}}.$$
(3.13)

Let  $N_0(r) = \left(\frac{c(n-2)}{bh(r)r^{n-2}}\right)^{\frac{1}{n-3}}$ . By continuity of h and since  $h(r_0) > 0$ , we can choose the numbers  $C_1, C_2$  and  $\delta$  in (3.8) and (3.10) such that

$$C_1 < N_0(r) < C_2$$

for any  $r \in (r_0 - \delta, r_0 + \delta)$ . By direct computation we get from Proposition 3.2.2 that  $\frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} > 0$  when  $\mu = (N_0(r) - \frac{1}{k^{\frac{n-2}{n-3}}})k^{\frac{n-2}{n-3}}$ , and  $\frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} < 0$  when  $\mu = (N_0(r) + \frac{1}{k^{\frac{n-2}{n-3}}})k^{\frac{n-2}{n-3}}$  for some  $\sigma > \theta > 0$  where  $\sigma$  is as in Proposition 3.2.2. Thus there exists  $\bar{\mu_k}(r) \in ((N_0(r) - \frac{1}{k^{\frac{n-2}{n-3}}})k^{\frac{n-2}{n-3}}, (N_0(r) + \frac{1}{k^{\frac{n-2}{n-3}}})k^{\frac{n-2}{n-3}})$  such that  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$  attains a local maximum point in  $\mu$  at  $\bar{\mu_k}(r)$ .

On the other hand, letting  $N_k(r) = \overline{\mu}_k(r)k^{\frac{2-n}{n-3}}$  and using (3.13), we have

$$\frac{bh(r)}{\bar{\mu_{k}}} - \frac{ck^{n-2}}{\bar{\mu_{k}}^{n-2}r^{n-2}} = \left(\frac{bh(r)}{N_{k}(r)} - \frac{c}{N_{k}(r)^{n-2}r^{n-2}}\right) \frac{1}{k^{\frac{n-2}{n-3}}} = \left(\frac{bh(r)}{N_{0}(r)} - \frac{c}{N_{0}(r)^{n-2}r^{n-2}} + O(|N_{k}(r) - N_{0}(r)|^{2})\right) \frac{1}{k^{\frac{n-2}{n-3}}} = \left(\frac{n-3}{n-2}\frac{bh(r)}{N_{0}(r)} + O(\frac{1}{\mu^{2\theta}_{0,k}})\right) \frac{1}{k^{\frac{n-2}{n-3}}} = (b'(rh(r))^{\frac{n-2}{n-3}} + O(\frac{1}{\mu^{2\theta}_{0,k}})) \frac{1}{k^{\frac{n-2}{n-3}}}$$

$$(3.14)$$

for some b' > 0 independent of k. So now if rh(r) has a local maximum point at  $r_0 > 0$ , it follows from (3.14) and Proposition 3.2.2 that there exists  $\bar{r}_k \in (r_0 - \delta, r_0 + \delta)$  such that  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$  attains a local maximum point at  $(\bar{r}_k, \bar{\mu}_k(\bar{r}_k))$ . In case rh(r) has a local minimum point at  $r_0 > 0$  with  $h(r_0) > 0$ , there exists  $\delta > 0$  such that

$$0 < r_0 h(r_0) < r h(r)$$

for all  $r \in [r_0 - \delta, r_0 + \delta]$ . Define a new variable  $t = \mu - N_0(r)k^{\frac{n-2}{n-3}}$  and

$$J_k(r,t) := -I(W_{k,r,t+N_0(r)k^{\frac{n-2}{n-3}}} + \Phi_{k,r,t+N_0(r)k^{\frac{n-2}{n-3}}}).$$

It follows from Proposition 3.2.2 that  $\frac{\partial J_k(r,t)}{\partial t} < 0$  when  $t = -k^{\frac{n-2}{n-3}(1-\theta)}$ , and  $\frac{\partial J_k(r,t)}{\partial \mu} > 0$  when  $t = k^{\frac{n-2}{n-3}(1-\theta)}$  for some  $0 < \theta < \sigma$  independent of k. By using (3.14) we have that

$$\inf_{t\in\Omega_2} (J_k(r_0,t)) > \sup_{(r,t)\in\partial\Omega_1\times\Omega_2} (J_k(r,t))$$

for k large and  $\Omega_1 = (r_0 - \delta, r_0 + \delta), \Omega_2 = (-k^{\frac{n-2}{n-3}(1-\theta)}, k^{\frac{n-2}{n-3}(1-\theta)})$ . Now by Lemma 3.2.3 we get that  $J_k(r, t)$  has a critical point in the interior of  $\Omega = \Omega_1 \times \Omega_2$ . It is easy to check that this is equivalent to the existence of a critical point for  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$ .

Now when n = 3, define  $\mu_k(s) := e^{sklnk}$  for s > 0. Then it follows from Proposition 3.2.2 that

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) := I_k(r,s) = ak + k^2 lnke^{-sklnk} (bh(r)s - \frac{c}{r} + o(1))$$
(3.15)

when k is large. Since rh(r) attains a strict local maximum point at  $r_0 > 0$  with  $h(r_0) > 0$ , there exists  $\delta > 0$  such that

$$r_0h(r_0) > rh(r) > 0$$

for all  $r \in [r_0 - \delta, r_0 + \delta]$ . Define  $J_k(r, t) := I_k(r, s_k(r) + t)$  where  $s_k(r, t) = \frac{c}{brh(r)} + \frac{1}{klnk}$ is the maximum point of  $e^{-sklnk}(bh(r)s - \frac{c}{r})$  in s. Using (3.15) we have

$$J_k(r,t) = ak + k^2 lnk e^{-(s_k(r)+t)klnk} (bh(r)t + o(1)).$$
(3.16)

Taking  $0 < t_0 < \min\{\frac{s_k(r_0)}{2}, \frac{2}{3}(s_k(r_0 + \delta) - s_k(r_0)), \frac{2}{3}(s_k(r_0 - \delta) - s_k(r_0))\}$ , it follows from (3.16) that

$$J_k(r, t_0) < J_k(r_0, \frac{t_0}{2}),$$
  
$$J_k(r, -t_0) < J_k(r_0, \frac{t_0}{2}),$$

and

$$J_k(r_0 \pm \delta, t) < J_k(r_0, \frac{t_0}{2})$$

for k large and  $(r, t) \in [r_0 - \delta, r_0 + \delta] \times [-t_0, t_0]$ . Thus we know that  $J_k$  has a local maximum point in  $(r, t) \in [r_0 - \delta, r_0 + \delta] \times [-t_0, t_0]$  for large k. This ends the proof of Theorem 3.1.2.

## 3.3 Finite dimensional reduction

We will prove Proposition 3.2.1 in this section. The problem (3.12) can be written as

$$L_{k,r,\mu}(\Phi) = Q_{k,r,\mu}(N_{k,r,\mu}(\Phi) + R_{k,r,\mu})$$
(3.17)

in  $P_{k,r,\mu}$  where  $L_{k,r,\mu}$  is as in (3.7),  $Q_{k,r,\mu}$  is the orthogonal projection,  $\Phi \in P_{k,r,\mu}$  and

$$N_{k,r,\mu}(\Phi) = i^*((W_{k,r,\mu} + \Phi)_+^{\frac{n}{n-2}}) - i^*(W_{k,r,\mu}^{\frac{n}{n-2}}) - i^*(\frac{n}{n-2}W^{\frac{2}{n-2}}\Phi),$$
(3.18)

$$R_{k,r,\mu} = i^* (W_{k,r,\mu}^{\frac{n}{n-2}}) - W_{k,r,\mu}.$$
(3.19)

First of all we will perform the estimate of  $N_{k,r,\mu}(\Phi)$  in Lemma 3.3.1 and the estimate of  $R_{k,r,\mu}$  in Lemma 3.3.2. Then we will prove  $L_{k,r,\mu}$  is an isomorphism in the projection space  $P_{k,r,\mu}$  in Lemma 3.3.3 and Lemma 3.3.4. After that we will complete the proof of Proposition 3.2.1. **Lemma 3.3.1.** Let  $\mu$ , r be as in (3.8), (3.9) and (3.10). There exists C > 0 independent of k such that

$$|\langle N_{k,r,\mu}(\Phi), v \rangle| \leqslant C \|\Phi\|^{\frac{n}{n-2}} \|v\|$$

when  $n \ge 4$ ,

$$|\langle N_{k,r,\mu}(\Phi), v \rangle| \leq Cmax(k^{\frac{1}{4}} ||\Phi||^2, ||\Phi||^3) ||v||$$

when n = 3 for any  $\Phi, v \in P_{k,r,\mu}$ .

*Proof.* By the definition of  $N_{k,r,\mu}(\Phi)$  we have that

$$\langle N_{k,r,\mu}(\Phi), v \rangle = \int_{\partial \mathbb{R}^n_+} ((W_{k,r,\mu} + \Phi)^{\frac{n}{n-2}}_+ - W^{\frac{n}{n-2}}_{k,r,\mu} - \frac{n}{n-2} W^{\frac{2}{n-2}}_{k,r,\mu} \Phi) v.$$

Since

$$|(W_{k,r,\mu} + \Phi)_{+}^{\frac{n}{n-2}} - W_{k,r,\mu}^{\frac{n}{n-2}} - \frac{n}{n-2}W_{k,r,\mu}^{\frac{2}{n-2}}\Phi| \leqslant C|\Phi|^{\frac{n}{n-2}}$$

when  $n \ge 4$ , and

$$|(W_{k,r,\mu} + \Phi)_{+}^{\frac{n}{n-2}} - W_{k,r,\mu}^{\frac{n}{n-2}} - \frac{n}{n-2}W_{k,r,\mu}^{\frac{2}{n-2}}\Phi| \leq Cmax(W_{k,r,\mu}|\Phi|^2, |\Phi|^3)$$

when n = 3, we have by Hölder and Sobolev inequalities that

$$\langle N_{k,r,\mu}(\Phi), v \rangle | \leq C \|\Phi\|^{\frac{n}{n-2}} \|v\|$$

when  $n \ge 4$ ,

$$|\langle N_{k,r,\mu}(\Phi), v \rangle| \leq Cmax(k^{\frac{1}{4}} \|\Phi\|^2, \|\Phi\|^3) \|v\|$$

when n = 3.

Now we estimate  $R_{k,r,\mu}$ .

**Lemma 3.3.2.** Let  $\mu$ , r be as in (3.8), (3.9) and (3.10). There exists C > 0 independent of k and  $k_0$  positive integer such that for  $k \ge k_0$ 

$$\|R_{k,r,\mu}\| \leqslant C\frac{k}{\mu}$$

when  $n \ge 5$ ,

$$\|R_{k,r,\mu}\| \leqslant C \frac{k \ln \mu}{\mu}$$

when n = 4,

$$\|R_{k,r,\mu}\| \leqslant C \frac{k}{\mu^{\frac{1}{2}}}$$

when n = 3.

*Proof.* For any  $v \in P_{k,r,\mu}$  we have by the definition of  $R_{k,r,\mu}$  that

$$\langle R_{k,r,\mu}, v \rangle = \int_{\mathbb{R}^{n}_{+}} (-DW_{k,r,\mu}Dv) + \int_{\partial\mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{n}{n-2}}v - hW_{k,r,\mu}v)$$
  
= 
$$\int_{\partial\mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{n}{n-2}} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{n}{n-2}})v + \int_{\partial\mathbb{R}^{n}_{+}} (-hW_{k,r,\mu}v)$$
  
= 
$$I_{1} + I_{2}.$$

Step 1: Estimate  $I_1$ .

Define

$$\Omega_j = \left\{ y = (y_1, y_2, y') : \left\langle \frac{(y_1, y_2)}{|(y_1, y_2)|}, \frac{x_j}{|x_j|} \right\rangle \ge \cos(\frac{\pi}{k}) \right\}.$$
(3.20)

Then we have

$$\int_{\partial \mathbb{R}^n_+} (W_{k,r,\mu}^{\frac{n}{n-2}} - \sum_{i=1}^k U_{x_i,\mu}^{\frac{n}{n-2}})v = k \int_{\Omega_1 \cap \partial \mathbb{R}^n_+} (W_{k,r,\mu}^{\frac{n}{n-2}} - \sum_{i=1}^k U_{x_i,\mu}^{\frac{n}{n-2}})v.$$

We have that in  $\Omega_1$ ,

$$|W_{k,r,\mu}^{\frac{n}{n-2}} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{n}{n-2}}| \leqslant C(U_{x_{1},\mu}^{\frac{2}{n-2}} \sum_{j=2}^{k} U_{x_{j},\mu} + (\sum_{j=2}^{k} U_{x_{j},\mu})^{\frac{n}{n-2}})$$

for some C > 0 and for  $j \neq 1$ ,

$$|y - x_j| \ge \frac{1}{2}|x_j - x_1|, |y - x_j| \ge |y - x_1|.$$

We also have on  $\partial \mathbb{R}^n_+$ 

$$U_{x_j,\mu} = C(\frac{\mu}{1+\mu^2|\bar{y}-\bar{x_j}|^2})^{\frac{n-2}{2}}.$$

Thus

$$U_{x_{1},\mu}^{\frac{2}{n-2}}U_{x_{j},\mu} \leqslant C\mu^{\frac{n}{2}} (\frac{1}{1+\mu^{2}|\bar{y}-\bar{x}_{1}|^{2}})^{\frac{n+\epsilon}{4}} (\frac{1}{\mu^{2}|\bar{x}_{j}-\bar{x}_{1}|^{2}})^{\frac{n-\epsilon}{4}}$$

for some  $\epsilon > 0$  small when  $n \ge 4$  to be fixed later and  $\epsilon = 1$  in case n = 3. Therefore

$$U_{x_{1},\mu}^{\frac{2}{n-2}}\sum_{j=2}^{k}U_{x_{j},\mu}\leqslant C\mu^{\frac{n}{2}}(\frac{1}{1+\mu^{2}|\bar{y}-\bar{x_{1}}|^{2}})^{\frac{n+\epsilon}{4}}\sum_{j=2}^{k}(\frac{1}{\mu|\bar{x_{j}}-\bar{x_{1}}|})^{\frac{n-\epsilon}{2}}$$

for some C > 0. By Hölder and Sobolev inequalities we have

$$\left|\int_{\Omega_{1}\cap\partial\mathbb{R}^{n}_{+}} U_{x_{1},\mu}^{\frac{2}{n-2}} \sum_{j=2}^{k} U_{x_{j},\mu}v\right| \leqslant C \sum_{j=2}^{k} \left(\frac{1}{\mu|\bar{x}_{j}-\bar{x}_{1}|}\right)^{\frac{n-\epsilon}{2}} \frac{1}{k^{\frac{n-2}{2(n-1)}}} \|v\|$$

for some C > 0. Similarly we have for some C > 0,

$$U_{x_j,\mu} \leqslant \frac{C}{(\mu|\bar{x_j} - \bar{x_1}|)^{\frac{n-2}{2} - \frac{(n-2)\epsilon}{2n}}} \frac{\mu^{\frac{n-2}{2}}}{(1 + \mu|\bar{y} - \bar{x_1}|)^{\frac{n-2}{2} + \frac{(n-2)\epsilon}{2n}}}.$$

 $\operatorname{So}$ 

$$\left(\sum_{j=2}^{k} U_{x_{j},\mu}\right)^{\frac{n}{n-2}} \leqslant C\left(\sum_{j=2}^{k} \frac{1}{(\mu|\bar{x_{j}}-\bar{x_{1}}|)^{\frac{n-2}{2}-\frac{(n-2)\epsilon}{2n}}}\right)^{\frac{n}{n-2}} \frac{\mu^{\frac{n}{2}}}{(1+\mu|\bar{y}-\bar{x_{1}}|)^{\frac{n+\epsilon}{2}}}.$$

Therefore

$$\int_{\Omega_1 \cap \partial \mathbb{R}^n_+} (\sum_{j=2}^k U_{x_j} \mu)^{\frac{n}{n-2}} v \leqslant C \frac{1}{k^{\frac{n-2}{2(n-1)}}} (\sum_{j=2}^k \frac{1}{(\mu |\bar{x_j} - \bar{x_1}|)^{\frac{n-2}{2} - \frac{(n-2)\epsilon}{2n}}})^{\frac{n}{n-2}} \|v\|$$

for some C > 0. When  $n \ge 4$ , taking  $0 < \epsilon < \frac{2}{(n-1)(n-2)}$ , we get that

$$|I_1| \leq Ck(\frac{k^{\frac{\epsilon}{2}}}{k^{\frac{n-2}{2(n-1)}}}(\frac{k}{\mu})^{\frac{n-\epsilon}{2}})||v||$$

when  $n \ge 4$  and

$$|I_1| \leqslant Ck^{\frac{3}{4}}(\frac{k^3}{\mu}) \|v\|$$

when n = 3 for some C > 0 independent of k.

Step 2: Estimate of  $I_2$ .

By the symmetry condition of  $h, W_{k,r,\mu}$  and  $P_{k,r,\mu}$  we know

$$|I_2| = \left| \int_{\partial \mathbb{R}^n_+} hW_{k,r,\mu} v \right| = k \left| \int_{\partial \mathbb{R}^n_+} hU_{x_1,\mu} v \right|.$$

On one hand, we have by  $h \in L^2(\mathbb{R}^2)$  and Hölder inequality that

$$\left|\int_{\partial \mathbb{R}^{n}_{+} \setminus B_{1}(x_{1})} h U_{x_{1},\mu} v\right| \leq \frac{C}{\mu^{\frac{n-2}{2}}} \|v\|$$

for some C > 0. On the other hand, again by Hölder inequality, we have for some C > 0

$$\left|\int_{B_{1}(x_{1})} hU_{x_{1},\mu}v\right| \leqslant C\left(\int_{B_{1}(x_{1})} U_{x_{1},\mu}^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} \|v\|.$$

By direct calculation we have

$$\left(\int_{B_1(x_1)} U_{x_1,\mu}^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = O(\frac{1}{\mu})$$

when  $n \ge 5$ ,

$$\left(\int_{B_1(x_1)} U_{x_1,\mu}^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = O\left(\frac{\ln\mu}{\mu}\right)$$

when n = 4 and

$$\left(\int_{B_1(x_1)} U_{x_1,\mu}^{\frac{2(n-1)}{n}}\right)^{\frac{n}{2(n-1)}} = O\left(\frac{1}{\mu^{\frac{1}{2}}}\right)$$

when n = 3. Combining all these above we have that

$$|I_2| = k | \int_{\partial \mathbb{R}^n_+} h U_{x_1,\mu} v | \leqslant C \frac{k}{\mu^{\frac{1}{2}}} ||v|$$

when n = 3,

$$|I_2| \leqslant C \frac{k l n \mu}{\mu} \|v\|$$

when n = 4,

$$|I_2| \leqslant C \frac{k}{\mu} \|v\|$$

when  $n \ge 5$  for some C > 0 independent of k. Using the estimate of  $I_1$  and  $I_2$ , we conclude the proof of Lemma 3.3.2.

Now we want to show that  $L_{k,r,\mu}$  is an isomorphism in  $P_{k,r,\mu}$ .

**Lemma 3.3.3.** Let  $L_{k,r,\mu}$  be as in (3.7). There exists C > 0 and  $k_0 > 0$  such that for every  $k > k_0$  we have

$$\|L_{k,r,\mu}\Phi\| \ge C\|\Phi\|$$

for every  $\Phi \in P_{k,r,\mu}$ .

*Proof.* For any u, v in  $P_{k,r,\mu}$  we have that

$$\langle L_{k,r,\mu}u,v\rangle = \langle u,v\rangle - \left\langle i^*\left(\frac{n}{n-2}W_{k,r,\mu}^{\frac{2}{n-2}}u\right),v\right\rangle$$
  
=  $\int_{\mathbb{R}^n_+} DuDv + \int_{\partial\mathbb{R}^n_+} (h_kuv - \frac{n}{n-2}W_{k,r,\mu}^{\frac{2}{n-2}}uv).$ 

Assume that for any k there exists  $\Phi_k \in P_{k,r,\mu}, r_k, \mu_k$  such that

$$\|\Phi_k\| = \sqrt{k}, \|L_{k,r,\mu}\Phi_k\| = o(\sqrt{k}),$$

Thus

$$\int_{\Omega_1} |D\Phi_k|^2 + \int_{\partial \mathbb{R}^n_+ \cap \Omega_1} h\Phi_k^2 = 1$$
(3.21)

and

$$\int_{\Omega_1} D\Phi_k Dv + \int_{\partial \mathbb{R}^n_+ \cap \Omega_1} (h\Phi_k v - \frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_k v) = o(1)$$
(3.22)

where  $\Omega_1$  is as in (3.20) for any  $v \in P_{k,r,\mu}$ . We now apply the rescaling

$$\widetilde{\Phi_k}(y) = \mu_k^{-\frac{n-2}{2}} \Phi_k(\mu_k^{-1}y + x_1).$$

Then from (3.21) we get that  $\widetilde{\Phi_k}$  is bounded in  $D^{1,2}(\mathbb{R}^n_+)$ . So by (3.3) we may assume that  $\widetilde{\Phi_k}$  converges weakly to  $\Phi$  in  $D^{1,2}(\mathbb{R}^n_+)$ ,  $L^{\frac{2(n-1)}{n-2}}(\partial \mathbb{R}^n_+)$  and strongly in  $L^2_{loc}(\mathbb{R}^n_+)$ . We define

$$\langle u, v \rangle_{\Omega_1} = \int_{\Omega_1} Du Dv + \int_{\partial \mathbb{R}^n_+ \cap \Omega_1} huv$$

and

$$||u||_{\Omega_1} = \int_{\Omega_1} |Du|^2 + \int_{\partial \mathbb{R}^n_+ \cap \Omega_1} hu^2$$

for all  $u, v \in P_{k,r,\mu}$ . Now by applying the rescaling to

$$\left\langle \sum_{i=1}^{k} Z_{i,r_k,\mu_k,j}, \Phi_k \right\rangle_{\Omega_1} = 0$$

j = 1, 2, and let k go to infinity we get that

$$0 = \int_{\mathbb{R}^{n}_{+}} D \frac{\partial U_{0,1}}{\partial x_{1}} D\Phi = \int_{\partial \mathbb{R}^{n}_{+}} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \frac{\partial U_{0,1}}{\partial x_{1}} \Phi$$
(3.23)

and

$$0 = \int_{\mathbb{R}^{n}_{+}} D \frac{\partial U_{0,\mu}}{\partial \mu}|_{\mu=1} D\Phi = \int_{\partial \mathbb{R}^{n}_{+}} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \frac{\partial U_{0,\mu}}{\partial \mu}|_{\mu=1} \Phi.$$
(3.24)

So now by the definition of  $L_{k,r,\mu}$  we have that there exists  $c_{1,k}, c_{2,k} \in \mathbb{R}$  such that

$$\Phi_k - i^* \left(\frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_k\right) - L_{k,r,\mu} \Phi_k = \sum_{j=1}^2 c_{j,k} \sum_{i=1}^k Z_{i,r_k,\mu_k,j}.$$
(3.25)

Apply  $\sum_{l=1}^{k} Z_{l,r_k,\mu_k,1}, \sum_{l=1}^{k} Z_{l,r_k,\mu_k,2}$  on (3.25), rescale and pass to the limit. We get that

$$\sum_{j=1}^{2} c_{j,k} \left\langle \sum_{i=1}^{k} Z_{i,r_{k},\mu_{k},j}, \sum_{l=1}^{k} Z_{l,r_{k},\mu_{k},1} \right\rangle_{\Omega_{1}} = \int_{\partial \mathbb{R}^{n}_{+}} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \frac{\partial U_{0,1}}{\partial x_{1}} \Phi + o(1) \quad (3.26)$$

and

$$\sum_{j=1}^{2} c_{j,k} \left\langle \sum_{i=1}^{k} Z_{i,r_{k},\mu_{k},j}, \sum_{l=1}^{k} Z_{l,r_{k},\mu_{k},2} \right\rangle_{\Omega_{1}} = \int_{\partial \mathbb{R}^{n}_{+}} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \frac{\partial U_{0,\mu}}{\partial \mu}|_{\mu=1} \Phi + o(1). \quad (3.27)$$

Moreover,

$$\left\langle \sum_{i=1}^{k} Z_{i,r_k\mu_k,j}, \sum_{i=1}^{k} Z_{i,r_k\mu_k,l} \right\rangle_{\Omega_1} = C_j \delta_{j,l} + o(1)$$
(3.28)

for some constant  $C_j > 0$  independent of k. By using (3.23), (3.24), (3.26), (3.27) and (3.28) we have

$$\begin{split} \|\Phi_{k} - i^{*} \left(\frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_{k}\right)\|_{\Omega_{1}} \\ &= O(\|L_{k,r,\mu} \Phi_{k}\|_{\Omega_{1}} + \sum_{j=1}^{2} |c_{j,k}|\| \sum_{i=1}^{k} Z_{i,r_{k},\mu_{k},j}\|_{\Omega_{1}}) = o(1). \end{split}$$
(3.29)

So for all  $v \in P_{k,r,\mu}$  we get

$$\begin{split} \langle \Phi_k, v \rangle_{\Omega_1} &= \left\langle \Phi_k - i^* (\frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_k) + i^* (\frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_k), v \right\rangle_{\Omega_1} \\ &= \int_{\partial \mathbb{R}^n_+ \cap \Omega_1} \frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_k v + o(1) = \int_{\partial \mathbb{R}^n_+} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \Phi \tilde{v} + o(1). \end{split}$$

On the other hand,

$$\langle \Phi_k, v \rangle_{\Omega_1} = \int_{\mathbb{R}^n_+} D\Phi D\tilde{v} + o(1).$$
 (3.30)

Thus we know  $\Phi$  is a solution of

$$\begin{cases} -\Delta \Phi = 0 \text{ in } \mathbb{R}^n_+ \\ \frac{\partial \Phi}{\partial \nu} = \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \Phi \text{ on } \partial \mathbb{R}^n_+. \end{cases}$$
(3.31)

It is proved by S.Almaraz [2] that the set of solutions to this linearized problem (3.31) is formed by a basis of  $\left\{\frac{\partial U_{0,1}}{\partial x_i}, \frac{\partial U_{0,\mu}}{\partial \mu}|_{\mu=1}\right\}$ . By the symmetry of  $\Phi$  with respect to  $y_2, ..., y_{n-1}$  and (3.23),(3.24), we get that  $\Phi = 0$  almost everywhere. Now by the convergence properties of  $\widetilde{\Phi_k}$  we have

$$\|\Phi_k\|_{\Omega_1}^2 = \int_{\partial \mathbb{R}^n_+ \cap \tilde{\Omega_1}} \frac{n}{n-2} U_{0,1}^{\frac{2}{n-2}} \widetilde{\Phi_k}^2 + o(1) = o(1)$$
(3.32)

 $(\tilde{\Omega}_1 \text{ comes from } \Omega_1 \text{ by rescaling})$  which is a contradiction to (3.21). Thus we have proven the lemma.

Now by Lemma 3.3.3 and the Fredholm alternative, we have:

**Lemma 3.3.4.** There exists  $k_0 > 0$  such that for every  $k \ge k_0$ ,  $L_{k,r,\mu}$  is an isomorphism on  $P_{k,r,\mu}$ .

We can now prove Proposition 3.2.1.

Proof of Proposition 3.2.1. When  $n \ge 5$ , let

$$V = \left\{ v \in P_{k,r,\mu} : \|v\| \leqslant \frac{C_0 k}{\mu} \right\}$$

for some  $C_0 > 0$  to be determined later. Now (3.17) is equivalent to

$$\Phi = T(\Phi) = L_{k,r,\mu}^{-1}(Q_{k,r,\mu}(N_{k,r,\mu}(\Phi) + R_{k,r,\mu})).$$

By Lemma 3.3.4 we know that there exists  $k_0 > 0$  such that for  $k \ge k_0$ ,

$$||T(\Phi)|| \leq C(||N_{k,r,\mu}(\Phi)|| + ||R_{k,r,\mu}||)$$

for some C > 0 independent of k. When  $n \ge 5$ , it follows from lemmas 3.3.1 and 3.3.2 that for all  $\Phi \in V$ ,

$$\begin{aligned} \|T(\Phi)\| &\leq C(\|N_{k,r,\mu}(\Phi)\| + \|R_{k,r,\mu}\|) \\ &\leq C(\|\Phi\|^{\frac{n}{n-2}} + \frac{k}{\mu}) \\ &\leq C_0(\frac{k}{\mu}) \end{aligned}$$

for some  $C_0 > 0$ . Since  $\mu \in [C_1 k^{\frac{n-2}{n-3}}, C_2 k^{\frac{n-2}{n-3}}]$ , choosing  $C_0$  from previous inequality, we see that T is a map from V to V. Now we want to show that T is a contraction map. By direct computation we have that for  $\Phi_1, \Phi_2 \in V$ ,

$$|\langle N_{k,r,\mu}(\Phi_1) - N_{k,r,\mu}(\Phi_2), v \rangle|$$
  

$$\leq C(\int_{\partial \mathbb{R}^n_+} (|\Phi_1|^{\frac{2}{n-2}} + |\Phi_2|^{\frac{2}{n-2}})|\Phi_1 - \Phi_2||v|)$$
  

$$\leq C(\|\Phi_1\|^{\frac{2}{n-2}} + \|\Phi_2\|^{\frac{2}{n-2}})\|\Phi_1 - \Phi_2\|\|v\|$$

for some C > 0 and  $v \in P_{k,r,\mu}$ . Thus by Lemma 3.3.4, there exists  $k'_0 > 0$  such that for  $k \ge k'_0$ 

$$||T(\Phi_1) - T(\Phi_2)|| \leq \frac{1}{2} ||\Phi_1 - \Phi_2||.$$

Thus we have proven that T is a contraction map. When n = 4 we pick

$$V = \left\{ v \in P_{k,r,\mu} : \|v\| \leqslant \frac{C'_0 k l n \mu}{\mu} \right\}$$

and when n = 3, we pick

$$V = \left\{ v \in P_{k,r,\mu} : \|v\| \leqslant \frac{C_0''k}{\mu^{\frac{1}{2}}} \right\}$$

for some  $C'_0, C''_0 > 0$  and similarly we can show that T is a contraction map. Now by the contraction mapping theorem we prove that there exists a unique solution  $\Phi_{k,r,\mu}$ to (3.17) in V. Now if there exists a critical point  $(\bar{r}_k, \bar{\mu}_k)$  of  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$ , we have by (3.28) and the estimate of  $\Phi_{k,r,\mu}$  that

$$0 = \frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} (\bar{r}_{k}, \bar{\mu}_{k})$$

$$= \left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial (W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} \right\rangle (\bar{r}_{k}, \bar{\mu}_{k})$$

$$= \left\langle \sum_{j=1}^{2} \sum_{i=1}^{k} c_{j,k} Z_{i,r,\mu,j}, \frac{\partial (W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} \right\rangle (\bar{r}_{k}, \bar{\mu}_{k})$$

$$= \frac{1}{\mu} \left\langle \sum_{j=1}^{2} \sum_{i=1}^{k} c_{j,k} Z_{i,r,\mu,j}, \sum_{i=1}^{k} Z_{i,r,\mu,2} \right\rangle (\bar{r}_{k}, \bar{\mu}_{k})$$

$$+ \left\langle \sum_{j=1}^{2} \sum_{i=1}^{k} c_{j,k} Z_{i,r,\mu,j}, \frac{\partial \Phi_{k,r,\mu}}{\partial \mu} \right\rangle (\bar{r}_{k}, \bar{\mu}_{k})$$

$$= \frac{c_{1,k}}{\mu} \left\langle \sum_{i=1}^{k} Z_{i,r,\mu,1}, \sum_{i=1}^{k} Z_{i,r,\mu,2} \right\rangle + \frac{c_{2,k}}{\mu} \left\langle \sum_{i=1}^{k} Z_{i,r,\mu,2}, \sum_{i=1}^{k} Z_{i,r,\mu,2} \right\rangle$$

$$+ \left\langle \sum_{j=1}^{2} \sum_{i=1}^{k} c_{j,k} \frac{\partial Z_{i,r,\mu,j}}{\partial \mu}, \Phi_{k,r,\mu} \right\rangle$$

$$= c_{1,k}o(\frac{k}{\mu}) + c_{2,k}(C_{2}\frac{k}{\mu} + o(\frac{k}{\mu})) + o(\frac{k}{\mu}(|c_{1,k}| + |c_{2,k}|)). \quad (3.33)$$

for some  $C_2$  independent of k and  $c_{1,k}, c_{2,k}$  depending on k. Similarly we have

$$0 = \frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial r} (\bar{r_k}, \bar{\mu_k})$$
  
=  $c_{2,k}o(\mu k) + c_{1,k}(C_1\mu k + o(\mu k)) + o(\mu k(|c_{1,k}| + |c_{2,k}|)).$  (3.34)

Thus from (3.33) and (3.34) we have

$$c_{1,k} + c_{2,k} + o(|c_{1,k}| + |c_{2,k}|) = 0.$$
(3.35)

From (3.35) we know

$$c_{1,k} = c_{2,k} = 0$$

for k large which means  $W_{k,\bar{r_k},\bar{\mu_k}} + \Phi_{k,\bar{r_k},\bar{\mu_k}}$  is a solution to

$$\begin{cases} \Delta u = 0 \ in \ \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} + hu = u_+^{\frac{n}{n-2}} \ on \ \partial \mathbb{R}^n_+. \end{cases}$$

By using the coercivity condition we obtain that  $W_{k,\bar{r_k},\bar{\mu_k}} + \Phi_{k,\bar{r_k},\bar{\mu_k}} \ge 0$  a.e. in  $\partial \mathbb{R}^n_+$ . Thus  $W_{k,\bar{r_k},\bar{\mu_k}} + \Phi_{k,\bar{r_k},\bar{\mu_k}}$  is a positive solution to (3.1).

## 3.4 Energy expansion

We perform the estimate of  $I(W_{k,r,\mu} + \Phi_{k,r,\mu})$  in this section where  $W_{k,r,\mu}$  is as in (3.6) and  $\Phi_{k,r,\mu} \in P_{k,r,\mu}$  is the unique solution we obtained in Proposition 3.2.1. Recall that

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^n_+} |Du|^2 + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} hu^2 - \frac{(n-2)}{2(n-1)} \int_{\partial \mathbb{R}^n_+} u_+^{\frac{2(n-1)}{n-2}}.$$

**Lemma 3.4.1.** Let  $\mu$  be as in (3.8) and (3.9). We have

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = I(W_{k,r,\mu}) + O(\frac{k}{\mu^{1+\sigma}})$$

when  $n \ge 4$  and

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = I(W_{k,r,\mu}) + O(\frac{k^2}{\mu})$$

when n=3 for some  $\sigma > 0$  independent of k.

*Proof.* Since

$$DI(W_{k,r,\mu} + \Phi_{k,r,\mu})(\Phi_{k,r,\mu}) = \langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \Phi_{k,r,\mu} \rangle = 0$$

we know when  $n \ge 4$  there exists  $0 \le t_1, t_2 \le 1$  such that

$$\begin{split} &I(W_{k,r,\mu} + \Phi_{k,r,\mu}) \\ &= I(W_{k,r,\mu}) + DI(W_{k,r,\mu} + t_1 \Phi_{k,r,\mu})(\Phi_{k,r,\mu}) \\ &= I(W_{k,r,\mu}) + (t_1 - 1)D^2 I(W_{k,r,\mu} + t_2 \Phi_{k,r,\mu})(\Phi_{k,r,\mu}, \Phi_{k,r,\mu}) \\ &= I(W_{k,r,\mu}) + O(\int_{\mathbb{R}^n_+} (|D\Phi_{k,r,\mu}|^2) + \int_{\partial\mathbb{R}^n_+} (h\Phi_{k,r,\mu}^2 - \frac{n}{n-2}(W_{k,r,\mu} + \Phi_{k,r,\mu})^{\frac{2}{n-2}}\Phi_{k,r,\mu}^2)) \\ &= I(W_{k,r,\mu}) + O(||\Phi_{k,r,\mu}||^2 + ||\Phi_{k,r,\mu}||^{\frac{2(n-1)}{n-2}} + \int_{\partial\mathbb{R}^n_+} W_{k,r,\mu}^{\frac{2}{n-2}}\Phi_{k,r,\mu}^2). \end{split}$$

Since  $\Phi_{k,r,\mu} \in P_k$  is the solution to (3.17), using (3.7) we have

$$\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \Phi_{k,r,\mu} \rangle = \|\Phi_{k,r,\mu}\|^2 - \frac{n}{n-2} \int_{\partial \mathbb{R}^n_+} W_{k,r,\mu}^{\frac{2}{n-2}} \Phi_{k,r,\mu}^2$$
$$= \langle N_{k,r,\mu}(\Phi_{k,r,\mu}) + R_{k,r,\mu}, \Phi_{k,r,\mu} \rangle .$$

Thus

$$\left|\int_{\partial\mathbb{R}^{n}_{+}}W_{k,r,\mu}^{\frac{2}{n-2}}\Phi_{k,r,\mu}^{2}\right| = O(\|\Phi_{k,r,\mu}\|^{2} + (\|N_{k,r,\mu}(\Phi_{k,r,\mu})\| + \|R_{k,r,\mu}\|)\|\Phi_{k,r,\mu}\|).$$

So from Proposition 3.2.1 we get

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = I(W_{k,r,\mu}) + O(\frac{k}{\mu^{1+\sigma}})$$

when  $n \ge 4$  and

$$I(W_{k,r,\mu} + \Phi_{k,r,\mu}) = I(W_{k,r,\mu}) + O(\frac{k^2}{\mu}).$$

when n = 3 for some  $\sigma > 0$  independent of k.

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**Lemma 3.4.2.** Let  $\mu, r$  be as in (3.8), (3.9) and (3.10). We have

$$\frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} = \left\langle I'(W_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle + O(\frac{k}{\mu^{2+\sigma}})$$

when  $n \ge 4$  for some  $\sigma > 0$  independent of k.

Proof. Now

$$\begin{aligned} \frac{\partial I(W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} &= \left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial (W_{k,r,\mu} + \Phi_{k,r,\mu})}{\partial \mu} \right\rangle \\ &= \left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle \\ &+ \left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial \Phi_{k,r,\mu}}{\partial \mu} \right\rangle \\ &= I_1 + I_2. \end{aligned}$$

Step 1: Estimate of  $I_1$ .

$$I_{1} = \int_{\mathbb{R}^{n}_{+}} \left( D(W_{k,r,\mu} + \Phi_{k,r,\mu}) D \frac{\partial W_{k,r,\mu}}{\partial \mu} + \int_{\partial \mathbb{R}^{n}_{+}} (h(W_{k,r,\mu} + \Phi_{k,r,\mu})) \frac{\partial W_{k,r,\mu}}{\partial \mu} - (W_{k,r,\mu} + \Phi_{k,r,\mu})^{\frac{n}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \right)$$
$$= \left\langle I'(W_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle + \int_{\mathbb{R}^{n}_{+}} D \Phi_{k,r,\mu} D \frac{\partial W_{k,r,\mu}}{\partial \mu} + \int_{\partial \mathbb{R}^{n}_{+}} h \Phi_{k,r,\mu} \frac{\partial W}{\partial \mu} - \int_{\partial \mathbb{R}^{n}_{+}} ((W_{k,r,\mu} + \Phi_{k,r,\mu}))^{\frac{n}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} - W_{k,r,\mu}^{\frac{n}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu}).$$
(3.36)

When  $n \ge 4$ , by Hölder inequality we have

$$\int_{\partial \mathbb{R}^{n}_{+}} ((W_{k,r,\mu} + \Phi_{k,r,\mu})^{\frac{n}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} - W_{k,r,\mu}^{\frac{n}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu}) \\
= \frac{n}{n-2} \int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} + O(\int_{\partial \mathbb{R}^{n}_{+}} |\Phi_{k,r,\mu}|^{\frac{n}{n-2}} |\frac{\partial W_{k,r,\mu}}{\partial \mu}|) \\
= \frac{n}{n-2} \int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} \\
+ O(\frac{\|\Phi_{k,r,\mu}\|^{\frac{n}{n-2}}}{\mu} (\int_{\partial \mathbb{R}^{n}_{+}} |W_{k,r,\mu}|^{\frac{2(n-1)}{n-2}})^{\frac{n-2}{2(n-1)}}) \\
= \frac{n}{n-2} \int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} + O(\frac{k\|\Phi_{k,r,\mu}\|^{\frac{n}{n-2}}}{\mu}).$$
(3.37)

Since  $\Phi_{k,r,\mu} \in P_{k,r,\mu}$ , we get

$$0 = \int_{\mathbb{R}^{n}_{+}} D \frac{\partial W_{k,r,\mu}}{\partial \mu} D \Phi_{k,r,\mu} + \int_{\partial \mathbb{R}^{n}_{+}} h \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu}$$
(3.38)

which gives

$$0 = \sum_{i=1}^{k} \left( \int_{\mathbb{R}^{n}_{+}} D \frac{\partial U_{x_{i},\mu}}{\partial \mu} D \Phi_{k,r,\mu} + \int_{\partial \mathbb{R}^{n}_{+}} h \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} \right)$$
$$= \int_{\partial \mathbb{R}^{n}_{+}} \left( \frac{n}{n-2} \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} + h \sum_{i=1}^{k} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} \right).$$
(3.39)

Combining (3.36)-(3.39), we get

$$I_{1} = \left\langle I'(W_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle - \int_{\partial \mathbb{R}^{n}_{+}} \left( \frac{n}{n-2} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} - \frac{n}{n-2} \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} - h \sum_{i=1}^{k} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} \right) + O\left(\frac{k \|\Phi_{k,r,\mu}\|^{\frac{n}{n-2}}}{\mu}\right).$$

$$(3.40)$$

Now we estimate the terms in (3.40) above. The estimates (3.41) and (3.42) are obtained similarly as in the proof of Lemma 3.3.2 except here we replace v by  $\Phi_{k,r,\mu}$ and we use the result of Proposition 3.2.1. When  $n \ge 4$ ,

$$\begin{split} |\int_{\partial \mathbb{R}^{n}_{+}} h \sum_{i=1}^{k} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu}| &= k |\int_{\partial \mathbb{R}^{n}_{+}} h \frac{\partial U_{x_{1},\mu}}{\partial \mu} \Phi_{k,r,\mu}| \\ &\leqslant \frac{Ck}{\mu} \int_{\partial \mathbb{R}^{n}_{+}} |h| |U_{x_{1},\mu}| |\Phi_{k,r,\mu}| \\ &\leqslant \frac{Ck ln\mu}{\mu^{2}} ||\Phi_{k,r,\mu}|| \\ &\leqslant \frac{Ck}{\mu^{2+\sigma}} \end{split}$$
(3.41)

and

$$\begin{split} &|\int_{\mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{2} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu})| \\ &= k |\int_{\Omega_{1}} (W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu})| \\ &\leqslant \frac{Ck}{\mu} \int_{\Omega_{1}} |W_{k,r,\mu}^{\frac{n}{n-2}} \Phi_{k,r,\mu} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{n}{n-2}} \Phi_{k,r,\mu}| \\ &\leqslant \frac{Ck}{\mu^{2+\sigma}} \end{split}$$
(3.42)

for some C > 0 and  $\sigma > 0$  independent of k where  $\Omega_1$  is as in (3.20). Now from (3.41), (3.42) and Proposition 3.2.1 we have

$$\left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle = \left\langle I'(W_{k,r,\mu}), \frac{\partial W_{k,r,\mu}}{\partial \mu} \right\rangle + O(\frac{k}{\mu^{2+\sigma}})$$
(3.43)

for some  $\sigma > 0$  independent of k.

Step 2: Estimate of  $I_2$ .

It follows from (3.7), (3.18) and (3.19) that

$$\left\langle I'(W_{k,r,\mu} + \Phi_{k,r,\mu}), \frac{\partial \Phi_{k,r,\mu}}{\partial \mu} \right\rangle = \sum_{j=1}^{2} c_{j,k} \left\langle \sum_{i=1}^{k} Z_{i,r,\mu,j}, \frac{\partial \Phi_{k,r,\mu}}{\partial \mu} \right\rangle$$

for some  $c_{j,k}$  constants depending on k. Here  $L_{k,r,\mu}$ ,  $N_{k,r,\mu}$ ,  $R_{k,r,\mu}$  are as in (3.17), (3.18) and (3.19),  $Z_{i,r,\mu,j}$  are as in (3.4) and (3.5). We have by (3.17) that

$$L_{k,r,\mu}(\Phi_{k,r,\mu}) - N_{k,r,\mu}(\Phi_{k,r,\mu}) - R_{k,r,\mu} = \sum_{j=1}^{2} c_{j,k} \sum_{i=1}^{k} Z_{i,r,\mu,j}$$
(3.44)

for some  $c_{j,k}$  and since  $\Phi_{k,r,\mu} \in P_{k,r,\mu}$ ,

$$\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,j} \right\rangle = \left\langle L_{k,r,\mu}(\sum_{i=1}^{k} Z_{i,r,\mu,j}), \Phi_{k,r,\mu} \right\rangle$$
$$= -\frac{n}{n-2} \int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \sum_{i=1}^{k} Z_{i,r,\mu,j} \Phi_{k,r,\mu}.$$
(3.45)

By (3.4), (3.5) and (3.45) and we have,

$$\left|\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,1} \right\rangle\right| = O\left(\frac{1}{\mu} \left| \int_{\partial \mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial r} \Phi_{k,r,\mu}) \right|\right)$$
(3.46)

and

$$\left|\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,2} \right\rangle\right| = O(\mu \left| \int_{\partial \mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu}) \right|)$$
(3.47)

Since  $\Phi_{k,r,\mu} \in P_{k,r,\mu}$ , we obtain

$$\int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} \\
= \int_{\partial \mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial \mu} \Phi_{k,r,\mu} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu} \\
- \frac{n-2}{n} h \sum_{i=1}^{k} \frac{\partial U_{x_{i},\mu}}{\partial \mu} \Phi_{k,r,\mu})$$
(3.48)

and

$$\int_{\partial \mathbb{R}^{n}_{+}} W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W}{\partial r} \Phi_{k,r,\mu} 
= \int_{\partial \mathbb{R}^{n}_{+}} (W_{k,r,\mu}^{\frac{2}{n-2}} \frac{\partial W_{k,r,\mu}}{\partial r} \Phi_{k,r,\mu} - \sum_{i=1}^{k} U_{x_{i},\mu}^{\frac{2}{n-2}} \frac{\partial U_{x_{i},\mu}}{\partial r} \Phi_{k,r,\mu} 
- \frac{n-2}{n} h \sum_{i=1}^{k} \frac{\partial U_{x_{i},\mu}}{\partial r} \Phi_{k,r,\mu}).$$
(3.49)

By using (3.41), (3.42), (3.46)-(3.49) and analogue estimates for  $\frac{\partial W_{k,r,\mu}}{\partial r}$  we obtain when  $n \ge 4$ ,

$$\left|\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,1} \right\rangle\right| = O(\frac{k}{\mu^{2+\sigma}})$$
(3.50)

and

$$\left|\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,2} \right\rangle\right| = O(\frac{k}{\mu^{1+\sigma}})$$
(3.51)

for some  $\sigma>0$  independent of k. Since

$$c_{j,k} \left\langle \sum_{i=1}^{k} Z_{i,r,\mu,j}, \sum_{i=1}^{k} Z_{i,r,\mu,j} \right\rangle = \left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}) - N_{k,r,\mu}(\Phi_{k,r,\mu}) - R_{k,r,\mu}, \sum_{i=1}^{k} Z_{i,r,\mu,j} \right\rangle,$$
(3.52)

we have that

$$|c_{j,k}| = O(\frac{\left|\left\langle L_{k,r,\mu}(\Phi_{k,r,\mu}), \sum_{i=1}^{k} Z_{i,r,\mu,j}\right\rangle\right|}{\|\sum_{i=1}^{k} Z_{i,r,\mu,j}\|^{2}} + \frac{(\|N_{k,r,\mu}(\Phi_{k,r,\mu})\| + \|R_{k,r,\mu}\|)\|\sum_{i=1}^{k} Z_{i,r,\mu,j}\|}{\|\sum_{i=1}^{k} Z_{i,r,\mu,j}\|^{2}}).$$
(3.53)

By combining (3.28), (3.50), (3.51), (3.53), Proposition 3.2.1 and Lemmas 3.3.1 and 3.3.2 we get that when  $n \ge 4$ 

$$\begin{aligned} |I_2| &= |\sum_{j=1}^2 c_{j,k} \sum_{i=1}^k \left\langle Z_{i,r,\mu,j}, \frac{\partial \Phi_{k,r,\mu}}{\partial \mu} \right\rangle | \\ &= |\sum_{j=1}^2 c_{j,k} \sum_{i=1}^k \left\langle \Phi_{k,r,\mu}, \frac{\partial Z_{i,r,\mu,j}}{\partial \mu} \right\rangle | \\ &= O(|\sum_{j=1}^2 |c_{j,k}| ||\Phi_{k,r,\mu}|| \sum_{i=1}^k \frac{\partial Z_{i,r,\mu,j}}{\partial \mu} ||) \\ &= O(\frac{k}{\mu} (\sum_{j=1}^2 |c_{j,k}| ||\Phi_{k,r,\mu}||)) \\ &= O(\frac{k}{\mu^{2+\sigma}}) \end{aligned}$$

for some  $\sigma > 0$  independent of k. This ends the proof of Lemma 3.4.2.

We perform the energy expansion in Lemma 3.4.3 below.

**Lemma 3.4.3.** Let  $r, \mu$  be as in (3.8), (3.9) and (3.10). We have

$$I(W_{k,r,\mu}) = k\left(a + \frac{bh(r)}{\mu} - \frac{ck^{n-2}}{\mu^{n-2}r^{n-2}} + O(\frac{1}{\mu^{1+\sigma}})\right),$$
$$\frac{\partial I(W_{k,r,\mu})}{\partial \mu} = k\left(-\frac{bh(r)}{\mu^2} + \frac{c(n-2)k^{n-2}}{\mu^{n-1}r^{n-2}} + O(\frac{1}{\mu^{2+\sigma}})\right)$$

when  $n \ge 4$ , and

$$I(W_{k,r,\mu}) = k(a + \frac{bh(r)ln\mu}{\mu} - \frac{cklnk}{\mu r} + O(\frac{k}{\mu}))$$

when n=3 for some  $\sigma, a, b, c > 0$  constants independent of k.

*Proof.* By symmetry we know

$$\begin{split} \int_{\mathbb{R}^{n}_{+}} |DW_{k,r,\mu}|^{2} &= \int_{\mathbb{R}^{n}_{+}} -\Delta W_{k,r,\mu} W_{k,r,\mu} + \int_{\partial\mathbb{R}^{n}_{+}} \frac{\partial W_{k,r,\mu}}{\partial\nu} W_{k,r,\mu} \\ &= \int_{\partial\mathbb{R}^{n}_{+}} \sum_{i=1}^{k} U_{x_{i,\mu}}^{\frac{n}{n-2}} W_{k,r,\mu} = \sum_{i=1}^{k} \sum_{j=1}^{k} (\int_{\partial\mathbb{R}^{n}_{+}} U_{x_{i,\mu}}^{\frac{n}{n-2}} U_{x_{j,\mu}}) \\ &= k (\int_{\partial\mathbb{R}^{n}_{+}} U_{0,1}^{\frac{2(n-1)}{n-2}} + \sum_{i=2}^{k} \int_{\partial\mathbb{R}^{n}_{+}} U_{x_{1,\mu}}^{\frac{n}{n-2}} U_{x_{i,\mu}}) \\ &= k (\int_{\partial\mathbb{R}^{n}_{+}} U_{0,1}^{\frac{2(n-1)}{n-2}} + \sum_{i=2}^{k} \frac{B_{0}}{\mu^{n-2}|x_{i}-x_{1}|^{n-2}} + O(\sum_{i=2}^{k} \frac{1}{\mu^{n-2+\sigma}|x_{i}-x_{1}|^{n-2+\sigma}})) \end{split}$$
(3.54)

for some  $B_0, \sigma > 0$  indepedent of k. Again by symmetry,

$$\begin{split} \int_{\partial \mathbb{R}^{n}_{+}} |W_{k,r,\mu}^{\frac{2(n-1)}{n-2}}| &= k \int_{\Omega_{1} \cap \partial \mathbb{R}^{n}_{+}} |W_{k,r,\mu}^{\frac{2(n-1)}{n-2}}| \\ &= k (\int_{\Omega_{1} \cap \partial \mathbb{R}^{n}_{+}} U_{x_{1},\mu}^{\frac{2(n-1)}{n-2}} + \frac{2(n-1)}{n-2} \int_{\Omega_{1} \cap \partial \mathbb{R}^{n}_{+}} \sum_{i=2}^{k} U_{x_{i},\mu}^{\frac{n-2}{n-2}} U_{x_{1},\mu} \\ &+ O(\int_{\Omega_{1} \cap \partial \mathbb{R}^{n}_{+}} (\sum_{i=2}^{k} U_{x_{1},\mu})^{\frac{n-1}{n-2}} U_{x_{i},\mu}^{\frac{n-1}{n-2}})) \\ &= k (\int_{\mathbb{R}^{n}_{+}} U_{0,1}^{\frac{2(n-1)}{n-2}} + \frac{2(n-1)}{n-2} \sum_{i=2}^{k} \frac{B_{0}}{\mu^{n-2} |x_{i} - x_{1}|^{n-2}} \\ &+ O(\sum_{i=2}^{k} \frac{1}{\mu^{n-2+\sigma} |x_{i} - x_{1}|^{n-2+\sigma}})). \end{split}$$
(3.55)
Moreover, we have

$$\int_{\partial \mathbb{R}^{n}_{+}} hW_{k,r,\mu}^{2} = k(\int_{\partial \mathbb{R}^{n}_{+}} hU_{x_{1},\mu}^{2} + O(\int_{\partial \mathbb{R}^{n}_{+}} h\sum_{i=2}^{k} U_{x_{1},\mu}U_{x_{i},\mu})).$$
(3.56)

Since  $h \in L^{n-1}(\mathbb{R}^n_+)$ , we get that:

$$\int_{\partial \mathbb{R}^n_+} h U^2_{x_1,\mu} = \frac{Ch(r)}{\mu} + O(\frac{1}{\mu^{1+\sigma}})$$
(3.57)

and

$$\int_{\partial \mathbb{R}^n_+} h U_{x_1,\mu} U_{x_i,\mu} = O(\frac{1}{\mu^{n-2} |x_1 - x_i|^{n-3}})$$
(3.58)

when  $n \ge 4$ ,

$$\int_{\mathbb{R}^2} h U_{x_1,\mu}^2 = \frac{Ch(r)ln\mu}{\mu} + O(\frac{1}{\mu})$$
(3.59)

and

$$\int_{\mathbb{R}^2} h U_{x_1,\mu} U_{x_i,\mu} = O(\frac{\ln|x_1 - x_i|}{\mu})$$
(3.60)

when n = 3 for  $i \neq 1$  and some  $C, \sigma > 0$  independent of k. Combining (3.54)-(3.60) and noting that

$$\sum_{i=2}^{k} \frac{1}{|x_i - x_1|^n} = \frac{Ck^n}{r^n} + O(k^{n-1})$$
(3.61)

when  $n \ge 2$ ,

$$\sum_{i=2}^{k} \frac{1}{|x_i - x_1|} = \frac{Cklnk}{r} + O(k)$$
(3.62)

and

$$\sum_{i=2}^{k} \ln|x_i - x_1| = O(k) \tag{3.63}$$

we have

$$I(W_{k,r,\mu}) = k(a + \frac{bh(r)}{\mu} - \frac{ck^{n-2}}{\mu^{n-2}r^{n-2}} + O(\frac{1}{\mu^{1+\sigma}}))$$
(3.64)

when  $n \ge 4$ , and

$$I(W_{k,r,\mu}) = k(a + \frac{bh(r)ln\mu}{\mu} - \frac{cklnk}{\mu r} + O(\frac{k}{\mu}))$$
(3.65)

when n = 3 for some  $a, b, c, \sigma > 0$  independent of k. Applying similar arguments to  $\frac{\partial I(W_{k,r,\mu})}{\partial \mu} \text{ when } n \ge 4 \text{ we get that}$ 

$$\frac{\partial I(W_{k,r,\mu})}{\partial \mu} = k(-\frac{bh(r)}{\mu^2} + \frac{c(n-2)k^{n-2}}{\mu^{n-1}r^{n-2}} + O(\frac{1}{\mu^{2+\sigma}}))$$
(3.66)

when  $n \ge 4$  for some  $a, b, c, \sigma > 0$  independent of k.

Now we can finish the proof of Proposition 3.2.2.

Proof of Proposition 3.2.2. Now Proposition 3.2.2 is a direct corollary of Lemmas 3.4.1-3.4.3.

# CHAPTER 4 A COMPACTNESS RESULT

In this chapter, we prove a compactness result to a boundary Yamabe problem in dimension n = 3. This chapter is based on the following paper:

• Sérgio Almaraz, Olivaine S. de Queiroz and Shaodong Wang, A compactness theorem for scalar-flat metrics on 3-manifolds with boundary, J. Funct. Anal. (2018).

### 4.1 Introduction

Let (M, g) be a Riemannian n-dimensional manifold with boundary  $\partial M$ , and let  $\nabla$  be its Riemannian connection. Denote by  $R_g$  its scalar curvature and by  $\Delta_g$ its Laplace-Beltrami operator, which is the Hessian trace. By  $h_g$  we denote the boundary mean curvature with respect to the inward normal vector  $\eta$ , i.e.,  $h_g = \frac{1}{n-1} \sum_{i=1}^{n-1} g(\nabla_{e_i} e_i, \eta)$  for any orthonormal frame  $\{e_i\}_{i=1}^{n-1}$  of  $\partial M$ .

In this paper we study the question of compactness of the full set of positive solutions to the equations

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + K u^p = 0, & \text{on } \partial M, \end{cases}$$

$$(4.1)$$

where 1 and <math>K > 0 is a constant. Here,  $L_g = \Delta_g - \frac{n-2}{4(n-1)}R_g$  is the conformal Laplacian and  $B_g = \frac{\partial}{\partial \eta} - \frac{n-2}{2}h_g$  is the conformal boundary operator.

These equations have a very interesting geometrical meaning when  $p = \frac{n}{n-2}$ . A solution u > 0 of (4.1) represents a conformal metric  $\tilde{g} = u^{\frac{4}{n-2}}g$  with scalar curvature

 $R_{\tilde{g}} = 0$  and boundary mean curvature  $h_{\tilde{g}} = \frac{2}{n-2}K$ , as (4.1) becomes a particular case of the well known equations

$$\begin{cases} L_g u + \frac{n-2}{4(n-1)} R_{\tilde{g}} u^{\frac{n+2}{n-2}} = 0, & \text{in } M, \\ B_g u + \frac{n-2}{2} h_{\tilde{g}} u^{\frac{n}{n-2}} = 0, & \text{on } \partial M. \end{cases}$$

The existence of those metrics was first studied by Escobar [25] motivated by the classical Yamabe problem on closed manifolds. Regularity of solutions was obtained by Cherrier in [17].

The equations (4.1) have a variational formulation in terms of the functional

$$Q(u) = \frac{\int_{M} |\nabla_{g}u|^{2} + \frac{n-2}{4(n-1)} R_{g}u^{2} dv_{g} + \frac{n-2}{2} \int_{\partial M} h_{g}u^{2} d\sigma_{g}}{\left(\int_{\partial M} |u|^{p+1} d\sigma_{g}\right)^{\frac{2}{p+1}}} ,$$

where  $dv_g$  and  $d\sigma_g$  denote the volume forms of M and  $\partial M$ , respectively. A function u is a critical point for Q if and only if it solves (4.1). However, direct methods fail to work when  $p = \frac{n}{n-2}$ , as  $p + 1 = \frac{2(n-1)}{n-2}$  is critical for the Sobolev trace embedding  $H^1(M) \hookrightarrow L^{p+1}(\partial M)$ . This functional has also a geometrical meaning for the critical exponent case as it becomes

$$Q(u) = \frac{\frac{n-2}{4(n-1)} \int_M R_{\tilde{g}} dv_{\tilde{g}} + \frac{n-2}{2} \int_{\partial M} h_{\tilde{g}} d\sigma_{\tilde{g}}}{\operatorname{area}_{\tilde{g}}(\partial M)^{\frac{n-2}{n}}}.$$

Defining the conformal invariant

$$Q(M, \partial M) = \inf \left\{ Q(u); \ u \in C^1(\bar{M}), u \neq 0 \text{ on } \partial M \right\},\$$

Escobar [26] observed that, when finite,  $Q(M, \partial M)$  has the same sign of the first eigenvalue  $\lambda_1(B_g)$  of the problem

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + \lambda u = 0, & \text{on } \partial M. \end{cases}$$

If  $\lambda_1(B_g) < 0$ , the solution of the equations (4.1) is unique. If  $\lambda_1(B_g) = 0$ , the equations (4.1) become linear and the solutions are unique up to a multiplication by a positive constant. Hence, the only interesting case is the positive one.

When M is conformally equivalent to the unit ball  $B^n$ , the solutions of (4.1) are well known. The only nontrivial examples occur when  $p = \frac{n}{n-2}$  and they all represent metrics isometric to the Euclidean one [24]. In his case, the conformal diffeomorphisms of the ball produces a blowing-up family of solutions to (4.1).

Working in dimension n = 3, our main result extends to the general case the work of Felli and Ould Ahmedou [29], that established compactness of the set of solutions to (4.1) when  $\partial M$  is umbilic.

**Theorem 4.1.1.** Let (M, g) be a Riemannian 3-manifold with boundary  $\partial M$ . Suppose that  $Q(M, \partial M) > 0$  and M is not conformally equivalent to the unit ball. Then, given a small  $\gamma_0 > 0$ , there exists  $C(M, g, \gamma_0) > 0$  such that for any  $p \in \left[1 + \gamma_0, \frac{n}{n-2}\right]$  and any solution u > 0 of (4.1) we have

$$C^{-1} \le u \le C$$
 and  $||u||_{C^{2,\alpha}(M)} \le C$ ,

for some  $0 < \alpha < 1$ .

The subcritical Sobolev exponents  $p < \frac{n}{n-2}$  in Theorem 4.1.1 provide a connection with the linear case. Although we omit the argument (see [1,28,29,33]), a proof of existence of a solution to Escobar's problem [25] can be achieved by computing the Leray-Schauder degree of all solutions of equations (4.1).

In the case of manifolds without boundary, the question of compactness of the full set of smooth solutions to the Yamabe equation was first raised by R. Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold  $M^n$  is not conformally equivalent to the sphere  $S^n$ . This problem was studied in [20, 21, 38, 39, 42, 43, 48, 51] and was completely solved in [11, 13, 36]. In [11], Brendle discovered the first smooth counterexamples for dimensions  $n \geq 52$  (nonsmooth examples were obtained by Ambrosetti and Malchiodi in [9]). In [36], Khuri, Marques and Schoen proved compactness for dimensions  $3 \leq n \leq 24$ . Their proof contains both a local and a global aspect. The local aspect involves the vanishing of the Weyl tensor up to order  $\left[\frac{n-6}{2}\right]$  at any blow-up point and the global aspect involves the positive mass theorem. Finally, in [13], Brendle and Marques extended the counterexamples of [11] to the remaining dimensions  $25 \leq n \leq 51$ . In the case of nonempty umbilical boundary, the same compactness and noncompactness results were obtained by Disconzi and Khuri in [19] for the boundary condition  $B_q u = 0$ .

Despite its additional technical difficulties, the question of compactness of the solutions of (4.1) turns out to have great similarity with the one above for the classical Yamabe equation. In [28] Felli and Ould Ahmedou prove compactness for locally conformally flat manifolds with umbilic boundary, a result previously obtained by Schoen [48] for the classical Yamabe equation. In [2] the first author proves

the vanishing of the trace-free boundary second fundamental form at any blow-up point, a result inspired by the vanishing of the Weyl tensor obtained by Li-Zhang and Marques independently in [38, 39, 43]. On the other hand, the noncompactness results of Brendle and Marques inspired the first author's paper [3] which provides counterexamples in dimensions  $n \ge 25$  to compactness in (4.1). So Theorem 4.1.1 ensures that there is a critical dimension  $3 < n_0 \le 25$  such that compactness for the set of positive smooth solutions of (4.1) holds for  $n < n_0$  and fails for  $n \ge n_0$ .

Although the corresponding result for the classical Yamabe equation in dimension 3 was obtained by Li and Zhu in [42], our approach to Theorem 4.1.1 makes use of some further techniques of the later works [36, 43]. This is because the canonical bubble, coming from the Euclidean metric on  $B^3$ , fails to provide a good approximation for the blowing up solutions of (4.1).

The strategy of the proof of Theorem 4.1.1 is similar to the one proposed by Schoen in the case of manifolds without boundary. It is based on finding local obstructions to blow-up by means of a Pohozaev-type identity. Assuming that a sequence  $\{u_i\}$  of solutions has an isolated simple blow-up point, we approximate  $\{u_i\}$  by the standard Euclidean solution plus a correction term  $\phi_i$ . The function  $\phi_i$ is defined as a solution to a non-homogeneous linear equation and is similar to the one in [36]. We then use the Pohozaev identity to prove a local sign restriction in dimension three, which allows the reduction to the simple blow-up case. This sign restriction is used again to derive a contradiction with the positive mass theorem established in [4] for manifolds modeled on the Euclidean half-space.

A key point in dimension three is that this hypothesis simplifies the estimates on the right side of the Pohozaev identity as every geometric term, including  $\phi_i$ , only contributes to the high order terms in the proof of the local sign restriction. It contrasts with the case of higher dimensions where further estimates on the geometric terms would be needed. Another point that differs from the mentioned papers on compactness is that we only use a very rough control of the Green's function. The relation with the positive mass theorem comes from an integral expression obtained by Brendle-Chen in [12].

This paper is organized as follows. In Section 4.2 we present some preliminaries computations about the standard solution on the Euclidean half-space, Fermi coordinates and the conformal invariant equation associated to (4.1). The important Pohozaev identity and the mass term is studied in Section 4.3. The definition of isolated and isolated simple blow-up points and some additional properties are collected in Section 4.4, while the blow-up estimates are presented in Section 4.5. In Section 4.6 we come back to the Pohozaev integral and prova a sign restriction and consequences. Finally we give a proof of the main result in Section 4.7.

# 4.2 Preliminaries

### 4.2.1 Notations

Throughout this work we will make use of the index notation for tensors, commas denoting covariant differentiation. We will adopt the summation convention whenever confusion is not possible. When dealing with coordinates on manifolds with boundary, we will use indices  $1 \le i, j, k, l \le n - 1$  and  $1 \le a, b, c, d \le n$ . In this context, lines under or over an object mean the restriction of the metric to the boundary is involved. We will denote by g the Riemannian metric and set det  $g = \det g_{ab}$ . The induced metric on  $\partial M$  will be denoted by  $\bar{g}$ . We will denote by  $\nabla_g$  the covariant derivative and by  $\Delta_g$  the Laplacian-Beltrami operator. By  $R_g$  or R we will denote the scalar curvature. The second fundamental form of the boundary will be denoted by  $\pi_{kl}$  and the mean curvature,  $\frac{1}{n-1}tr(\pi_{kl})$ , by  $h_g$  or h.

By  $\mathbb{R}^n_+$  we will denote the half-space  $\{z = (z_1, ..., z_n) \in \mathbb{R}^n; z_n \ge 0\}$ . If  $z \in \mathbb{R}^n_+$  we set  $\overline{z} = (z_1, ..., z_{n-1}) \in \mathbb{R}^{n-1} \cong \partial \mathbb{R}^n_+$ . We define  $B^+_{\delta}(0) = \{z \in \mathbb{R}^n_+; |z| < \delta\}$ . We also denote  $B^+_{\delta} = B^+_{\delta}(0)$  for short. We set  $\partial^+ B^+_{\delta}(0) = \partial B^+_{\delta}(0) \cap \mathbb{R}^n_+ = \{z \in \mathbb{R}^n_+; |z| = \delta\}$ and  $\partial' B^+_{\delta}(0) = B^+_{\delta}(0) \cap \partial \mathbb{R}^n_+ = \{z \in \partial \mathbb{R}^n_+; |z| < \delta\}$ . Thus,  $\partial B^+_{\delta}(0) = \partial' B^+_{\delta}(0) \cup \partial^+ B^+_{\delta}(0)$ .

In various parts of the text, we will make use of Fermi coordinates (see Definition 4.2.2 below)

$$\psi: B^+_{\delta}(0) \to M$$

centered at a point  $x_0 \in \partial M$ . In this case, we will work in  $B^+_{\delta}(0) \subset \mathbb{R}^n_+$ .

# 4.2.2 Standard solutions in the Euclidean half-space

In this subsection we study the Euclidean Yamabe equation in  $\mathbb{R}^n_+$  and its linearization.

The simplest example of solution to the Yamabe-type problem we are concerned is the ball in  $\mathbb{R}^n$  with the canonical Euclidean metric. This ball is conformally equivalent to the half-space  $\mathbb{R}^n_+$  by the inversion  $F : \mathbb{R}^n_+ \to B^n \setminus \{(0, ..., 0, -1)\}$  with respect to the sphere with center (0, ..., 0, -1) and radius 1. Here,  $B^n$  is the Euclidean ball in  $\mathbb{R}^n$  with center (0, ..., 0, -1/2) and radius 1/2. The expression for F is

$$F(y_1, \dots, y_n) = \frac{(y_1, \dots, y_{n-1}, y_n + 1)}{y_1^2 + \dots + y_{n-1}^2 + (y_n + 1)^2} + (0, \dots, 0, -1),$$

and of course its inverse mapping  $F^{-1}$  has the same expression. An easy calculation shows that F is a conformal map and  $F^*g_{eucl} = U^{\frac{4}{n-2}}g_{eucl}$  in  $\mathbb{R}^n_+$ , where  $g_{eucl}$  is the Euclidean metric and  $U(y) = (y_1^2 + \ldots + y_{n-1}^2 + (y_n + 1)^2)^{-\frac{n-2}{2}}$ . The function U satisfies

$$\begin{cases} \Delta U = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial U}{\partial y_n} + (n-2)U^{\frac{n}{n-2}} = 0, & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$
(4.2)

Since the equations (4.2) are invariant by horizontal translations and scalings with respect to the origin, we obtain the following family of solutions of (4.2):

$$U_{\lambda,z}(y) = \left(\frac{\lambda}{(\lambda + y_n)^2 + \sum_{j=1}^{n-1} (y_j - z_j)^2}\right)^{\frac{n-2}{2}},$$
(4.3)

where  $\lambda > 0$  and  $z = (z_1, ..., z_{n-1}) \in \mathbb{R}^{n-1}$ .

In fact, the converse statement is also true: by a Liouville-type theorem in [41] (see also [18, 24]), any non-negative solution to the equations (4.2) is of the form (4.3) or is identically zero.

The existence of the family of solutions (4.3) has two important consequences. First, we see that the set of solutions of the equations (4.2) is non-compact. In particular, the set of solutions of (4.1) with  $p = \frac{n}{n-2}$  is not compact when  $M^n$  is conformally equivalent to  $B^n$ . Secondly, the functions  $\frac{\partial U}{\partial y_j}$ , for j = 1, ..., n - 1, and  $\frac{n-2}{2}U+y^b\frac{\partial U}{\partial y^b},$  are solutions to the following homogeneous linear problem:

$$\begin{cases} \Delta \psi = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial \psi}{\partial y_n} + nU^{\frac{2}{n-2}}\psi = 0, & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

$$(4.4)$$

**Lemma 4.2.1.** Suppose  $\psi$  is a solution to

$$\begin{cases} \Delta \psi = 0, & in \mathbb{R}^n_+ \\ \frac{\partial \psi}{\partial y_n} + nU^{\frac{2}{n-2}}\psi = 0, & on \partial \mathbb{R}^n_+. \end{cases}$$
(4.5)

If  $\psi(y) = O((1 + |y|)^{-\alpha})$  for some  $\alpha > 0$ , then there exist constants  $c_1, ..., c_n$  such that

$$\psi(y) = \sum_{i=1}^{n-1} c_i \frac{\partial U}{\partial y_j} + c_n \left(\frac{n-2}{2}U + y^b \frac{\partial U}{\partial y^b}\right)$$

*Proof.* This is [2, Lemma 2.1].

# 4.2.3 Coordinate expansions for the metric

Recall the definition of Fermi coordinates:

**Definition 4.2.2.** Let  $x_0 \in \partial M$  and choose boundary geodesic normal coordinates  $(z_1, ..., z_{n-1})$ , centered at  $x_0$ , of the point  $x \in \partial M$ . We say that  $z = (z_1, ..., z_n)$ , for small  $z_n \geq 0$ , are the *Fermi coordinates* (centered at  $x_0$ ) of the point  $\exp_x(z_n\eta(x)) \in M$ . Here, we denote by  $\eta(x)$  the inward unit normal vector to  $\partial M$  at x. In this case, we have a map  $\psi(z) = \exp_x(z_n\eta(x))$ , defined on a subset of  $\mathbb{R}^n_+$ .

It is easy to see that in these coordinates  $g_{nn} \equiv 1$  and  $g_{jn} \equiv 0$ , for j = 1, ..., n-1. The expansion for g in Fermi coordinates is given by:

$$g_{ij}(\psi(z)) = \delta_{ij} - 2\pi_{ij}(x_0)z_n + O(|z|^2),$$
  

$$g^{ij}(\psi(z)) = \delta_{ij} + 2\pi_{ij}(x_0)z_n + O(|z|^2).$$
(4.6)

The existence of conformal Fermi coordinates, introduced in [44], is stated as follows:

**Proposition 4.2.3.** For any given integer  $N \ge 1$ , there is a metric  $\tilde{g}$ , conformal to g, such that in  $\tilde{g}$ -Fermi coordinates  $\tilde{\psi} : B^+_{\delta}(0) \to M$  centered at  $x_0$ , we have

$$(\det \tilde{g})(\tilde{\psi}(z)) = 1 + O(|z|^N)$$

Moreover,  $\tilde{g}$  can be written as  $\tilde{g} = fg$ , where f is a positive function with  $f(x_0) = 1$ and  $\frac{\partial f}{\partial z_k}(x_0) = 0$  for k = 1, ..., n-1. In this metric we also have  $h(\tilde{\psi}(z)) = O(|z|^{N-1})$ .

*Proof.* The first part is [44, Proposition 3.1] and the last one follows from

$$h_g = \frac{-1}{2(n-1)} g^{ij} g_{ij,n} = \frac{-1}{2(n-1)} (\log \det g)_{,n} \,.$$

Remark 4.2.4. Since we are only handling the 3-dimensional case, in this paper we do not use Proposition 4.2.3 in its full generality. Indeed, N = 2 is enough for our purposes here, and this case could be easily obtained by assuming the vanishing of the boundary mean curvature.

#### 4.2.4 Conformal scalar and mean curvature equations

In this subsection we study the partial differential equation we will work with in the next sections:

$$\begin{cases} L_g u = 0, & \text{in } M, \\ B_g u + (n-2)f^{-\tau}u^p = 0, & \text{on } \partial M, \end{cases}$$

$$(4.7)$$

where  $\tau = \frac{n}{n-2} - p$ ,  $1 + \gamma_0 \le p \le \frac{n}{n-2}$  for some fixed  $\gamma_0 > 0$  and f is a positive function.

The equations (4.7) have an important scaling invariance property. Fix  $x_0 \in \partial M$ and take  $\delta > 0$  small, and consider Fermi coordinates  $\psi : B^+_{\delta}(0) \to M$  centered at  $x_0$ . Given s > 0 we define the renormalized function

$$v(y) = s^{\frac{1}{p-1}} u(\psi(sy)), \text{ for } y \in B^+_{\delta s^{-1}}(0).$$

Then

$$\begin{cases} L_{\hat{g}}v = 0, & \text{in } B_{\delta s^{-1}}^+(0), \\ B_{\hat{g}}v + (n-2)\hat{f}^{-\tau}v^p = 0, & \text{on } \partial' B_{\delta s^{-1}}^+(0), \end{cases}$$

where  $\hat{f}(y) = f(\psi(sy))$  and the metric  $\hat{g}$  is defined by  $\hat{g}_{kl}(y) = g_{kl}(\psi(sy))$ .

The reason to work with the equations (4.7), instead of (4.1), is that they have important conformal invariance properies. Suppose  $\tilde{g} = \zeta^{\frac{4}{n-2}}g$  is a metric conformal to g. It follows from the properties

$$L_{\zeta^{\frac{4}{n-2}}g}(\zeta^{-1}u) = \zeta^{-\frac{n+2}{n-2}}L_gu \quad \text{and} \quad B_{\zeta^{\frac{4}{n-2}}g}(\zeta^{-1}u) = \zeta^{-\frac{n}{n-2}}B_gu$$

that, if u is a solution of the equations (4.7), then  $\zeta^{-1}u$  satisfies

$$\begin{cases} L_{\tilde{g}}(\zeta^{-1}u) = 0, & \text{in } M, \\ B_{\tilde{g}}(\zeta^{-1}u) + (n-2)(\zeta f)^{-\tau}(\zeta^{-1}u)^p = 0, & \text{on } \partial M, \end{cases}$$

which are again equations of the same type.

**Notation.** Let  $\Omega \subset M$  be a domain in a Riemannian manifold (M, g). Let  $\{g_i\}$  be a sequence of metrics on  $\Omega$ . We say that  $u_i \in \mathcal{M}_i$  if  $u_i > 0$  satisfies

$$\begin{cases} L_{g_i} u_i = 0, & \text{in } \Omega, \\ B_{g_i} u_i + (n-2) f_i^{-\tau_i} u_i^{p_i} = 0, & \text{on } \Omega \cap \partial M, \end{cases}$$

$$(4.8)$$

where  $\tau_i = \frac{n}{n-2} - p_i$  and  $1 + \gamma_0 \le p_i \le \frac{n}{n-2}$  for some fixed  $\gamma_0 > 0$ .

In many parts of this article we will work with sequences  $\{u_i \in \mathcal{M}_i\}_{i=1}^{\infty}$ . In this case, we assume that  $f_i \to f$  in the  $C_{loc}^1$  topology, for some positive function f, and that  $g_i \to g_0$  in the  $C_{loc}^3$  topology, for some metric  $g_0$ .

By the conformal invariance stated above, we are allowed to replace the metric  $g_i$  by  $\zeta_i^{\frac{4}{n-2}}g_i$  as long as we have control of the conformal factors  $\zeta_i$ . In this case, we replace the sequence  $\{u_i\}$  by  $\{\zeta_i^{-1}u_i\}$ . In particular, we can use conformal Fermi coordinates (see Proposition 4.2.3) centered at some point  $x_i \in \partial M$ , as those conformal changes are uniformly controlled with respect to i by construction.

## 4.3 The Pohozaev identity and the mass term

Let g be a Riemannian metric on the half-ball  $B^+_{\delta}(0)$ . For any  $x = (x_1, ..., x_n) \in \mathbb{R}^n$  we set  $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$ . For any smooth function u on  $B^+_{\delta}(0)$  and

 $0<\rho<\delta$  we define

$$P(u,\rho) = \int_{\partial^+ B_{\rho}^+(0)} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |du|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma + \frac{\rho}{p+1} \int_{\partial (\partial' B_{\rho}^+(0))} K f^{-\tau} u^{p+1} d\bar{\sigma}$$

and

$$P'(u,\rho) = \int_{\partial^+ B_{\rho}^+(0)} \left( \frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |du|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma \, .$$

An integration by parts [2, Proposition 3.1] gives the following Pohozaev-type identity to be used in the analysis of blow-up sequences:

Proposition 4.3.1. If u is a solution of

$$\begin{cases} \Delta_g u - \frac{n-2}{4(n-1)} R_g u = 0, & \text{in } B_{\delta}^+(0), \\ \partial_n u - \frac{n-2}{2} h_g u + K f^{-\tau} u^p = 0, & \text{on } \partial' B_{\delta}^+(0), \end{cases}$$

where K is a constant, then

$$P(u,\rho) = -\int_{B^+_{\rho}(0)} \left( x^a \partial_a u + \frac{n-2}{2} u \right) A_g(u) dx + \frac{n-2}{2} \int_{\partial' B^+_{\rho}(0)} \left( \bar{x}^k \partial_k u + \frac{n-2}{2} u \right) h_g u d\bar{x} \\ - \frac{\tau}{p+1} \int_{\partial' B^+_{\rho}(0)} K(\bar{x}^k \partial_k f) f^{-\tau-1} u^{p+1} d\bar{x} + \left( \frac{n-1}{p+1} - \frac{n-2}{2} \right) \int_{\partial' B^+_{\rho}(0)} K f^{-\tau} u^{p+1} d\bar{x} \,,$$

where  $A_g = \Delta_g - \Delta - \frac{n-2}{4(n-1)}R_g$ . Here,  $\Delta$  stands for the Euclidean Laplacian.

While in Section 4.6 we will obtain a sign restriction for  $P'(u, \rho)$  by means of Proposition 4.3.1, in this section we handle  $P'(u, \rho)$  directly and relate it with a mass-type geometric invariant defined below. **Lemma 4.3.2.** If  $\phi(x) = u(x) - |x|^{2-n}$  then

$$P'(u,\rho) = \frac{n-2}{2} \int_{\partial^+ B_{\rho}^+(0)} \left( \frac{\partial}{\partial r} r^{2-n} \phi(x) - r^{2-n} \frac{\partial \phi}{\partial r} \right) d\sigma + \frac{1}{2} \int_{\partial^+ B_{\rho}^+(0)} \left( r \left( \frac{\partial \phi}{\partial r} \right)^2 - r |d\phi|^2 + \frac{\partial \phi}{\partial r} \left( (n-2)\phi + r \frac{\partial \phi}{\partial r} \right) \right) d\sigma .$$

*Proof.* Direct calculations give

$$\frac{n-2}{2}u\partial_{r}u - \frac{r}{2}|du|^{2} + r(\partial_{r}u)^{2} = \frac{1}{2}(\partial_{r}u)\left((n-2)u + r\partial_{r}u\right) + \frac{r}{2}\left((\partial_{r}u)^{2} - |du|^{2}\right)$$
$$= \frac{1}{2}\left((n-2)\partial_{r}r^{2-n}\phi - (n-2)r^{2-n}\partial_{r}\phi + (n-2)\phi\partial_{r}\phi + r(\partial_{r}\phi)^{2}\right)$$
$$+ \frac{r}{2}\left((\partial_{r}\phi)^{2} - |d\phi|^{2}\right),$$

from which the result follows.

**Definition 4.3.3.** Let (N, g) be a Riemannian manifold with a noncompact boundary  $\partial N$ . We say that N is asymptotically flat with order q > 0, if there is a compact set  $K \subset N$  and a diffeomorphism  $f : N \setminus K \to \mathbb{R}^n_+ \setminus \overline{B^+_1}$  such that, in the coordinate chart defined by f (which we call the asymptotic coordinates of N), we have

$$|g_{ab}(y) - \delta_{ab}| + |y||g_{ab,c}(y)| + |y|^2 |g_{ab,cd}(y)| = O(|y|^{-q}), \text{ as } |y| \to \infty,$$

where a, b, c, d = 1, ..., n.

Suppose the manifold N, of dimension  $n \ge 3$ , is asymptotically flat with order  $q > \frac{n-2}{2}$ , as defined above. Assume also that  $R_g$  is integrable on N, and  $h_g$  is integrable on  $\partial N$ . Let  $(y_1, ..., y_n)$  be the asymptotic coordinates induced by the

diffeomorphism f. Then the limit

$$m(g) = \lim_{R \to \infty} \left\{ \sum_{a,b=1}^{n} \int_{y \in \mathbb{R}^{n}_{+}, |y|=R} (g_{ab,b} - g_{bb,a}) \frac{y_{a}}{|y|} \, d\sigma + \sum_{i=1}^{n-1} \int_{y \in \partial \mathbb{R}^{n}_{+}, |y|=R} g_{ni} \frac{y_{i}}{|y|} \, d\sigma \right\}$$
(4.9)

exists, and we call it the mass of (M, g). As proved in [4], m(g) is a geometric invariant in the sense that it does not depend on the asymptotic coordinates.

The expression in (4.9) is due to F. Marques and is the analogue of the ADM mass for the manifolds of Definition 4.3.3. A positive mass theorem for m(g), similar to the classical ones in [50, 62], is stated as follows:

**Theorem 4.3.4** ([4]). Assume n = 3. If  $R_g$ ,  $h_g \ge 0$ , then we have  $m(g) \ge 0$  and the equality holds if and only if N is isometric to  $\mathbb{R}^3_+$ .

The asymptotically flat manifolds we work with in this paper come from the stereographic projection of compact manifolds with boundary. Inspired by Schoen's approach [47] to the classical Yamabe problem, this projection is defined by means of a Green's function with singularity at a boundary point. Since we do not have the control of the Green's function expression used in the case of manifolds without boundary, the relation with (4.9) is obtained by means of an integral defined in [12]. This is stated in the next proposition.

**Proposition 4.3.5.** Let (M, g) be a compact n-manifold with boundary and consider Fermi coordinates centered at  $x_0 \in \partial M$ . If  $d = \left[\frac{n-2}{2}\right]$ , suppose in those coordinates we have

$$g_{ab}(x) = \delta_{ab} + h_{ab}(x) + O(|x|^{2d+2})$$

with  $h_{ab}(x) = O(|x|^{d+1})$  and  $tr(h_{ab}(x)) = O(|x|^{2d+2})$ . Let G be a smooth postive function on  $M \setminus \{x_0\}$  written near  $x_0$  as

$$G(x) = |x|^{2-n} + \phi(x)$$

where  $\phi$  is smooth on  $M \setminus \{x_0\}$  satisfying  $\phi(x) = O(|x|^{d+3-n} |\log |x||)$ . If we define the metric  $\hat{g} = G^{\frac{4}{n-2}}g$  and set

$$I(x_{0},\rho) = \frac{4(n-1)}{n-2} \int_{\partial^{+}B_{\rho}^{+}(0)} \left( |x|^{2-n} \partial_{a}G(x) - \partial_{a}|x|^{2-n}G(x) \right) \frac{x_{a}}{|x|} d\sigma$$
$$- \int_{\partial^{+}B_{\rho}^{+}(0)} \left( |x|^{3-2n} x_{a} \partial_{b}h_{ab}(x) - 2n|x|^{1-2n} x_{a} x_{b}h_{ab}(x) \right) d\sigma \,,$$

then  $(M \setminus \{x_0\}, \hat{g})$  is asymptotically flat in the sense of Definition 4.3.3 with mass

$$m(\hat{g}) = \lim_{\rho \to 0} I(x_0, \rho).$$

*Proof.* Consider inverted coordinates  $y_a = |x|^{-2}x_a$ . The first statement follows from the fact that  $\hat{g}\left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b}\right) = \delta_{ab} + O(|y|^{-d-1}|\log|y||)$ . In order to prove the last one, we can mimic the proof of [12, Proposition 4.3] to obtain

$$\int_{\partial^+ B^+_{\rho^{-1}}(0)} \frac{y_a}{|y|} \frac{\partial}{\partial y_b} \hat{g}\left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b}\right) d\sigma_{\rho^{-1}} - \int_{\partial^+ B^+_{\rho^{-1}}(0)} \frac{y_a}{|y|} \frac{\partial}{\partial y_a} \hat{g}\left(\frac{\partial}{\partial y_b}, \frac{\partial}{\partial y_b}\right) d\sigma_{\rho^{-1}} = \mathcal{I}(x_0, \rho) + O(\rho^{2d+4-n}(\log \rho)^2) \,.$$

Since  $(x_1, ..., x_n)$  are Femi coordinates,

$$\hat{g}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_n}\right) = 0$$
, for  $i = 1, ..., n - 1$ , if  $y_n = 0$ 

the result then follows

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**Proposition 4.3.6.** If in addition to the hypotheses of Proposition 4.3.5 we assume n = 3 and  $h(x_0) = tr(\pi_{ij}(x_0)) = 0$ , then

$$P'(G,\rho) = -\frac{1}{16}I(x_0,\rho) + O(\rho|\log \rho|).$$

*Proof.* Observe that in dimension n = 3 we have d = 0 and so  $\phi(x) = O(|\log |x||)$ , and the expansion (4.6) gives

$$h_{ij}(x) = 2\pi_{ij}(x_0)x_n + O(|x|^2), \quad h_{na}(x) = 0, \quad \text{tr}(\pi_{ij}(x_0)) = 0.$$

Then

$$\int_{\partial^+ B_{\rho}^+(0)} \left( |x|^{3-2n} x_a \partial_b h_{ab}(x) - 2n|x|^{1-2n} x_a x_b h_{ab}(x) \right) d\sigma$$
  
=  $-12 \int_{\partial^+ B_{\rho}^+(0)} x_i x_j \pi_{ij}(x_0) x_n |x|^{-5} d\sigma + O(\rho) = O(\rho) ,$ 

since the integral involving  $x_i x_j \pi_{ij}(x_0)$  vanishes by symmetry. A direct calculation shows

$$\int_{\partial^+ B^+_{\rho}(0)} \left( |x|^{2-n} \partial_a G(x) - \partial_a |x|^{2-n} G(x) \right) \frac{x_a}{|x|} d\sigma$$
$$= \int_{\partial^+ B^+_{\rho}(0)} \left( |x|^{2-n} \partial_a \phi(x) - \partial_a |x|^{2-n} \phi(x) \right) \frac{x_a}{|x|} d\sigma \,,$$

and so

$$I(x_0, \rho) = 8 \int_{\partial^+ B_{\rho}^+(0)} \left( |x|^{2-n} \partial_a \phi(x) - \partial_a |x|^{2-n} \phi(x) \right) \frac{x_a}{|x|} d\sigma + O(\rho) \,.$$

On the other hand, using Lemma 4.3.2 we obtain

$$P'(G,\rho) = -\frac{1}{2} \int_{\partial^+ B_{\rho}^+(0)} \left( |x|^{2-n} \partial_a \phi(x) - \partial_a |x|^{2-n} \phi(x) \right) \frac{x_a}{|x|} d\sigma + O(\rho |\log \rho|)$$

and the result follows.

#### 4.4 Isolated and isolated simple blow-up points

In this section we briefly collect the definitions and main results of isolated and isolated simple blow-up sequences from [2, Section 4]. They are inspired by the corresponding ones for manifolds without boundary and are similar to the ones in [28,29].

**Definition 4.4.1.** Let  $\Omega \subset M$  be a domain in a Riemannian manifold (M, g). We say that  $x_0 \in \Omega \cap \partial M$  is a *blow-up point* for the sequence  $\{u_i \in \mathcal{M}_i\}_{i=1}^{\infty}$ , if there is a sequence  $\{x_i\} \subset \Omega \cap \partial M$  such that

- (1)  $x_i \to x_0;$
- (2)  $u_i(x_i) \to \infty;$
- (3)  $x_i$  is a local maximum of  $u_i|_{\partial M}$ .

Briefly we say that  $x_i \to x_0$  is a blow-up point for  $\{u_i\}$ . The sequence  $\{u_i\}$  is called a *blow-up sequence*.

**Convention** If  $x_i \to x_0$  is a blow-up point, we use  $g_i$ -Fermi coordinates

$$\psi_i: B^+_\delta(0) \to M$$

centered at  $x_i$  and work in  $B^+_{\delta}(0) \subset \mathbb{R}^n_+$ , for some small  $\delta > 0$ .

**Notation.** If  $x_i \to x_0$  is a blow-up point we set  $M_i = u_i(x_i), \epsilon_i = M_i^{-(p_i-1)}$ .

**Definition 4.4.2.** We say that a blow-up point  $x_i \to x_0$  is an *isolated* blow-up point for  $\{u_i\}$  if there exist  $\delta, C > 0$  such that

$$u_i(x) \le C d_{\bar{g}_i}(x, x_i)^{-\frac{1}{p_i - 1}}, \quad \text{for all } x \in \partial M \setminus \{x_i\}, \ d_{\bar{g}_i}(x, x_i) < \delta.$$

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Since Fermi coordinates are normal on the boundary, the above definition is equivalent to

$$u_i(\psi_i(z)) \le C|z|^{-\frac{1}{p_i-1}}, \quad \text{for all } z \in \partial' B^+_{\delta}(0) \setminus \{0\}.$$

$$(4.10)$$

This definition is invariant under renormalization. This follows from the fact that if  $v_i(y) = s^{\frac{1}{p_i-1}} u_i(\psi_i(sy))$ , then

$$u_i(\psi_i(z)) \le C|z|^{-\frac{1}{p_i-1}} \iff v_i(y) \le C|y|^{-\frac{1}{p_i-1}},$$

where z = sy.

Harnack inequalities give the following two lemmas:

**Lemma 4.4.3.** Let  $x_i \to x_0$  be an isolated blow-up point. Then  $\{u_i\}$  satisfies

$$u_i(\psi_i(z)) \le C |z|^{-\frac{1}{p_i-1}}, \text{ for all } z \in B^+_{\delta}(0) \setminus \{0\}.$$

**Lemma 4.4.4.** Let  $x_i \to x_0$  be an isolated blow-up point and  $\delta$  as in Definition 4.4.2. Then there exists C > 0 such that for any  $0 < s < \frac{\delta}{3}$  we have

$$\max_{B_{2s}^+(0)\setminus B_{s/2}^+(0)} (u_i \circ \psi_i) \le C \min_{B_{2s}^+(0)\setminus B_{s/2}^+(0)} (u_i \circ \psi_i).$$

The next proposition says that, in the case of an isolated blow-up point, the sequence  $\{u_i\}$ , when renormalized, converges to the standard Euclidean solution U.

**Proposition 4.4.5.** Let  $x_i \to x_0$  be an isolated blow-up point. We set

$$v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{-(p_i-1)}y), \quad \text{for } y \in B^+_{\delta M_i^{p_i-1}}(0).$$

Then given  $R_i \rightarrow \infty$  and  $\beta_i \rightarrow 0$ , after choosing subsequences, we have

(a)  $|v_i - U|_{C^2(B^+_{R_i}(0))} < \beta_i;$ (b)  $\lim_{i \to \infty} \frac{R_i}{\log M_i} = 0;$ (c)  $\lim_{i \to \infty} p_i = \frac{n}{n-2}.$ 

Remark 4.4.6. Let  $x_i \to x_0$  and consider a conformal change  $\zeta_i^{\frac{4}{n-2}}g_i$  of the metrics  $g_i$  (see the last paragraph of Section 4.2.4). Suppose the conformal factors  $\zeta_i > 0$  are uniformly bounded (above and below) with  $\zeta_i(x_i) = 1$  and  $\frac{\partial \zeta_i}{\partial z_k}(x_i) = 0$  for k = 1, ..., n - 1. Then, using Proposition 4.4.5, it is not difficult to see that  $x_i \to x_0$  is an isolated blow-up point for  $\{u_i\}$  if and only it is for  $\{\zeta_i^{-1}u_i\}$ . This is the case when we use conformal Fermi coordinates (see Proposition 4.2.3) centered at  $x_i$ .

The set of blow-up points is handled in the next proposition.

**Proposition 4.4.7.** Given small  $\beta > 0$  and large R > 0 there exist constants  $C_0, C_1 > 0$ , depending only on  $\beta$ , R and  $(M^n, g)$ , such that if u is solution of (4.7) and  $\max_{\partial M} u \ge C_0$ , then  $\frac{n}{n-2} - p < \beta$  and there exist  $x_1, ..., x_N \in \partial M$ ,  $N = N(u) \ge 1$ , local maxima of u, such that:

(1) If  $r_j = Ru(x_j)^{-(p-1)}$  for j = 1, ..., N, then  $\{D_{r_j}(x_j) \subset \partial M\}_{j=1}^N$  is a disjoint collection, where  $D_{r_j}(x_j)$  is the boundary metric ball.

(2) For each j = 1, ..., N,  $|u(x_j)^{-1}u(\bar{\psi}_j(z)) - U(u(x_j)^{p-1}z)|_{C^2(B^+_{2r_j}(0))} < \beta$ , where we are using Fermi coordinates  $\bar{\psi}_j : B^+_{2r_j}(0) \to M$  centered at  $x_j$ . (3) We have

$$u(x) d_{\bar{g}}(x, \{x_1, ..., x_N\})^{\frac{1}{p-1}} \le C_1, \text{ for all } x \in \partial M,$$
$$u(x_j) d_{\bar{g}}(x_j, x_k)^{\frac{1}{p-1}} \ge C_0, \text{ for any } j \ne k, j, k = 1, ..., N.$$

We now introduce the notion of an isolated simple blow-up point. If  $x_i \to x_0$  is an isolated blow-up point for  $\{u_i\}$ , for  $0 < r < \delta$ , set

$$\bar{u}_i(r) = \frac{2}{\sigma_{n-1}r^{n-1}} \int_{\partial^+ B_r^+(0)} (u_i \circ \psi_i) d\sigma_r \quad \text{and} \quad w_i(r) = r^{\frac{1}{p_i-1}} \bar{u}_i(r) \,.$$

Note that the definition of  $w_i$  is invariant under renormalization. More precisely, if  $v_i(y) = s^{\frac{1}{p_i-1}} u_i(\psi_i(sy))$ , then  $r^{\frac{1}{p_i-1}} \bar{v}_i(r) = (sr)^{\frac{1}{p_i-1}} \bar{u}_i(sr)$ .

**Definition 4.4.8.** An isolated blow-up point  $x_i \to x_0$  for  $\{u_i\}$  is *simple* if there exists  $\delta > 0$  such that  $w_i$  has exactly one critical point in the interval  $(0, \delta)$ .

Remark 4.4.9. Let  $x_i \to x_0$  be an isolated blow-up point and  $R_i \to \infty$ . Using Proposition 4.4.5 it is not difficult to see that, choosing a subsequence,  $r \mapsto r^{\frac{1}{p_i-1}}\bar{u}_i(r)$ has exactly one critical point in the interval  $(0, r_i)$ , where  $r_i = R_i M_i^{-(p_i-1)} \to 0$ . Moreover, its derivative is negative right after the critical point. Hence, if  $x_i \to x_0$ is isolated simple then there exists  $\delta > 0$  such that  $w'_i(r) < 0$  for all  $r \in [r_i, \delta)$ .

A basic result for isolated simple blow-up point is stated as follows:

**Proposition 4.4.10.** Let  $x_i \to x_0$  be an isolated simple blow-up point for  $\{u_i\}$ . Then there exist  $C, \delta > 0$  such that

(a)  $M_i u_i(\psi_i(z)) \leq C |z|^{2-n}$  for all  $z \in B^+_{\delta}(0) \setminus \{0\}$ ;

(b)  $M_i u_i(\psi_i(z)) \ge C^{-1}G_i(z)$  for all  $z \in B^+_{\delta}(0) \setminus B^+_{r_i}(0)$ , where  $G_i$  is the Green's function so that:

$$\begin{cases} L_{g_i}G_i = 0, & \text{in } B_{\delta}^+(0) \setminus \{0\}, \\ G_i = 0, & \text{on } \partial^+ B_{\delta}^+(0), \\ B_{g_i}G_i = 0, & \text{on } \partial' B_{\delta}^+(0) \setminus \{0\} \end{cases}$$

and  $|z|^{n-2}G_i(z) \to 1$ , as  $|z| \to 0$ . Here,  $r_i$  is defined as in Remark 4.4.9.

Remark 4.4.11. Suppose that  $x_i \to x_0$  is an isolated simple blow-up point for  $\{u_i\}$ . Set

$$v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{-(p_i-1)}y), \text{ for } y \in B^+_{M_i^{p_i-1}\delta}(0).$$

Then, as a consequence of Propositions 4.4.5 and 4.4.10, we see that  $v_i \leq CU$  in  $B^+_{\delta M_i^{p_1-1}}(0).$ 

We finally have the following estimate for  $\tau_i = \frac{n}{n-2} - p_i$ , which is proved using Proposition 4.3.1:

**Proposition 4.4.12.** Let  $x_i \to x_0$  be an isolated simple blow-up point for  $\{u_i\}$  and let  $\rho > 0$  be small. Then there exists C > 0 such that

$$\tau_{i} \leq \begin{cases} C\epsilon_{i}^{1-2\rho+o_{i}(1)}, & \text{for } n \geq 4, \\ C\epsilon_{i}^{1-2\rho+o_{i}(1)}\log(\epsilon_{i}^{-1}), & \text{for } n = 3. \end{cases}$$
(4.11)

### 4.5 Blow-up estimates

In this section we give a pointwise estimate for a blow-up sequence  $\{u_i\}$  in a neighborhood of an isolated simple blow-up point. Our estimates are obtained for dimension n = 3.

Let  $x_i \to x_0$  be an isolated simple blow-up point for the sequence  $\{u_i \in \mathcal{M}_i\}$ . We use conformal Fermi coordinates centered at  $x_i$ . Thus we will work with conformal metrics  $\tilde{g}_i = \zeta_i^{\frac{4}{n-2}} g_i$  and sequences  $\{\tilde{u}_i = \zeta_i^{-1} u_i\}$  and  $\{\tilde{\epsilon}_i\}$ , where  $\tilde{\epsilon}_i = \tilde{u}_i(x_i)^{-(p_i-1)} =$  $\epsilon_i$ , since  $\zeta_i(x_i) = 1$ . As observed in Remark 4.4.6,  $x_i \to x_0$  is still an isolated blow-up point for the sequence  $\{\tilde{u}_i\}$  and satisfies the same estimates of Proposition 4.4.10 (since we have uniform control on the conformal factors  $\zeta_i > 0$ , these estimates are preserved). Let  $\psi_i : B^+_{\delta'}(0) \to M$  denote the  $\tilde{g}_i$ -Fermi coordinates centered at  $x_i$ .

In order to simplify our notations, we will omit the simbols  $\tilde{}$  and  $\psi_i$  in the rest of this section. Thus, the metrics  $\tilde{g}_i$  will be denoted by  $g_i$  and points  $\psi_i(x) \in M$ , for  $x \in B^+_{\delta'}(0)$ , will be denoted simply by x. In particular,  $x_i = \psi_i(0)$  will be denoted by 0 and  $u_i \circ \psi_i$  by  $u_i$ .

Set  $v_i(y) = \epsilon_i^{\frac{1}{p_i-1}} u_i(\epsilon_i y)$  for  $y \in B^+_{\delta' \epsilon_i^{-1}} = B^+_{\delta' \epsilon_i^{-1}}(0)$ . We know that  $v_i$  satisfies

$$\begin{cases} L_{\hat{g}_i} v_i = 0, & \text{in } B^+_{\delta' \epsilon_i^{-1}}, \\ B_{\hat{g}_i} v_i + (n-2) \hat{f}_i^{-\tau_i} v_i^{p_i} = 0, & \text{on } \partial' B^+_{\delta' \epsilon_i^{-1}}, \end{cases}$$
(4.12)

where  $\hat{f}_i(y) = f_i(\epsilon_i y)$  and  $\hat{g}_i$  is the metric with coefficients  $(\hat{g}_i)_{kl}(y) = (g_i)_{kl}(\epsilon_i y)$ .

Let  $r \mapsto 0 \leq \chi(r) \leq 1$  be a smooth cut-off function such that  $\chi(r) \equiv 1$  for  $0 \leq r \leq \delta$  and  $\chi(r) \equiv 0$  for  $r > 2\delta$ . We set  $\chi_{\epsilon}(r) = \chi(\epsilon r)$ . Thus,  $\chi_{\epsilon}(r) \equiv 1$  for  $0 \leq r \leq \delta \epsilon^{-1}$  and  $\chi_{\epsilon}(r) \equiv 0$  for  $r > 2\delta \epsilon^{-1}$ .

Observing that  $tr(\pi_{kl}(0)) = 0$  holds due to Proposition 4.2.3, by [3, Proposition 5.1] for every *i* there is a solution  $\phi_i$  of

$$\begin{cases} \Delta \phi_i(y) = -2\chi_{\epsilon_i}(|y|)\epsilon_i \pi_{kl}(0)y_n(\partial_k \partial_l U)(y), & \text{for } y \in \mathbb{R}^n_+, \\ \partial_n \phi_i(\bar{y}) + nU^{\frac{2}{n-2}}\phi_i(\bar{y}) = 0, & \text{for } \bar{y} \in \partial \mathbb{R}^n_+, \end{cases}$$

$$(4.13)$$

where  $\Delta$  stands for the Euclidean Laplacian, satisfying

$$|\nabla^r \phi_i|(y) \le C\epsilon_i |\pi_{kl}(0)|(1+|y|)^{3-r-n}, \quad \text{for } y \in \mathbb{R}^n_+, \ r = 0, 1 \text{ or } 2,$$
(4.14)

$$\phi_i(0) = \frac{\partial \phi_i}{\partial y_1}(0) = \dots = \frac{\partial \phi_i}{\partial y_{n-1}}(0) = 0$$
(4.15)

and

$$\int_{\partial \mathbb{R}^{n}_{+}} U^{\frac{n}{n-2}}(\bar{y}) \phi_{i}(\bar{y}) \, d\bar{y} = 0 \,. \tag{4.16}$$

Assumption In the rest of this section, n = 3.

**Lemma 4.5.1.** There exist  $\delta, C > 0$  such that, for  $|y| \leq \delta \epsilon_i^{-1}$ ,

$$|v_i - U - \phi_i|(y) \le C \max\{\epsilon_i, \tau_i\}.$$

*Proof.* We consider  $\delta < \delta'$  to be chosen later and set

$$\Lambda_{i} = \max_{|y| \le \delta \epsilon_{i}^{-1}} |v_{i} - U - \phi_{i}|(y) = |v_{i} - U - \phi_{i}|(y_{i}),$$

for some  $|y_i| \leq \delta \epsilon_i^{-1}$ . From Remark 4.4.11 we know that  $v_i(y) \leq CU(y)$  for  $|y| \leq \delta \epsilon_i^{-1}$ . Hence, if there exists c > 0 such that  $|y_i| \geq c \epsilon_i^{-1}$ , then

$$\Lambda_i = |v_i - U - \phi_i|(y_i) \le C |y_i|^{2-n} \le C \epsilon_i^{n-2}.$$

This implies the stronger inequality  $|v_i - U - \phi_i|(y) \leq C \epsilon_i^{n-2} = C \epsilon_i$ , for  $|y| \leq \delta \epsilon_i^{-1}$ . Hence, we can suppose that  $|y_i| \leq \delta \epsilon_i^{-1}/2$ .

Suppose, by contradiction, the result is false. Then, choosing a subsequence if necessary, we can suppose that

$$\Lambda_i^{-1}\epsilon_i \to 0 \quad \text{and} \quad \Lambda_i^{-1}\tau_i \to 0.$$
 (4.17)

Define

$$w_i(y) = \Lambda_i^{-1}(v_i - U - \phi_i)(y), \quad \text{for } |y| \le \delta \epsilon_i^{-1}.$$

By the equations (4.2) and (4.12),  $w_i$  satisfies

$$\begin{cases} L_{\hat{g}_i} w_i = Q_i, & \text{in } B^+_{\delta \epsilon_i^{-1}}, \\ B_{\hat{g}_i} w_i + b_i w_i = \overline{Q}_i, & \text{on } \partial' B^+_{\delta \epsilon_i^{-1}}, \end{cases}$$
(4.18)

where

$$\begin{split} b_i &= (n-2)\hat{f}_i^{-\tau_i} \frac{v_i^{p_i} - (U+\phi_i)^{p_i}}{v_i - (U+\phi_i)},\\ Q_i &= -\Lambda_i^{-1} \big\{ (L_{\hat{g}_i} - \Delta)(U+\phi_i) + \Delta\phi_i \big\},\\ \overline{Q}_i &= -\Lambda_i^{-1} \big\{ (n-2)\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i} - (n-2)U^{\frac{n}{n-2}} - nU^{\frac{2}{n-2}}\phi_i - \frac{n-2}{2}h_{\hat{g}_i}(U+\phi_i) \big\}. \end{split}$$

Observe that, for any function u,

$$\begin{split} (L_{\hat{g}_i} - \Delta)u(y) &= (\hat{g}_i^{kl} - \delta^{kl})(y)\partial_k\partial_l u(y) + (\partial_k \hat{g}_i^{kl})(y)\partial_l u(y) \\ &\quad - \frac{n-2}{4(n-1)}R_{\hat{g}_i}(y)u(y) + \frac{\partial_k\sqrt{\det\hat{g}_i}}{\sqrt{\det\hat{g}_i}}\hat{g}_i^{kl}(y)\partial_l u(y) \\ &= (g_i^{kl} - \delta^{kl})(\epsilon_i y)\partial_k\partial_l u(y) + \epsilon_i(\partial_k g_i^{kl})(\epsilon_i y)\partial_l u(y) \\ &\quad - \frac{n-2}{4(n-1)}\epsilon_i^2R_{g_i}(\epsilon_i y)u(y) + O(\epsilon_i^N|y|^{N-1})\partial_l u(y) \,. \end{split}$$

Hence,

$$Q_{i}(y) = -\Lambda_{i}^{-1}(g_{i}^{kl} - \delta^{kl})(\epsilon_{i}y)\partial_{k}\partial_{l}(U + \phi_{i})(y) - \Lambda_{i}^{-1}\epsilon_{i}(\partial_{k}g_{i}^{kl})(\epsilon_{i}y)\partial_{l}(U + \phi_{i})(y) + \frac{n-2}{4(n-1)}\Lambda_{i}^{-1}\epsilon_{i}^{2}R_{g_{i}}(\epsilon_{i}y)(U + \phi_{i})(y) - \Lambda_{i}^{-1}\Delta\phi_{i}(y) + O\left(\Lambda_{i}^{-1}\epsilon_{i}^{N}|y|^{N-1}(1 + |y|)^{1-n}\right) = O\left(\Lambda_{i}^{-1}\epsilon_{i}^{N}(1 + |y|)^{N-n}\right) + O\left(\Lambda_{i}^{-1}\epsilon_{i}^{2}(1 + |y|)^{2-n}\right).$$
(4.19)

Observe that

$$(n-2)\hat{f}_{i}^{-\tau_{i}}(U+\phi_{i})^{p_{i}}-(n-2)U^{\frac{n}{n-2}}-nU^{\frac{2}{n-2}}\phi_{i}$$

$$=(n-2)\left(\hat{f}_{i}^{-\tau_{i}}(U+\phi_{i})^{p_{i}}-(U+\phi_{i})^{\frac{n}{n-2}}\right)+O(U^{\frac{4-n}{n-2}}\phi_{i}^{2})$$

$$=(n-2)\hat{f}_{i}^{-\tau_{i}}\left((U+\phi_{i})^{p_{i}}-(U+\phi_{i})^{\frac{n}{n-2}}\right)$$

$$+(n-2)(\hat{f}_{i}^{-\tau_{i}}-1)(U+\phi_{i})^{\frac{n}{n-2}}+O(U^{\frac{4-n}{n-2}}\phi_{i}^{2}).$$

Using

$$\begin{split} U^{\frac{4-n}{n-2}}\phi_i^2 &= O(\epsilon_i^2|\pi_{kl}(0)|^2(1+|y|)^{2-n}),\\ h_{\hat{g}_i}(U+\phi_i) &= O(\epsilon_i^N(1+|y|)^{N+1-n}),\\ \hat{f}_i^{-\tau_i}\left((U+\phi_i)^{p_i} - (U+\phi_i)^{\frac{n}{n-2}}\right) &= O(\tau_i(U+\phi_i)^{\frac{n}{n-2}}\log(U+\phi_i)) = O(\tau_i(1+|y|)^{1-n}),\\ (\hat{f}_i^{-\tau_i} - 1)(U+\phi_i)^{\frac{n}{n-2}} &= O(\tau_i\log(f_i)(U+\phi_i)^{\frac{n}{n-2}}) = O(\tau_i(1+|y|)^{-n}), \end{split}$$

where in the second line we used Proposition 4.2.3, we obtain

$$\bar{Q}_i(\bar{y}) = O\left(\Lambda_i^{-1}\epsilon_i^2(1+|\bar{y}|)^{2-n}\right) + O\left(\Lambda_i^{-1}\tau_i(1+|\bar{y}|)^{1-n}\right) \,. \tag{4.20}$$

Moreover,

$$b_i(y) \to nU^{\frac{2}{n-2}}, \quad \text{in } C^2_{loc}(\mathbb{R}^n_+),$$
(4.21)

and

$$b_i(y) \le C(1+|y|)^{-2}, \quad \text{for } |y| \le \delta \epsilon_i^{-1}.$$
 (4.22)

Since  $|w_i| \leq |w_i(y_i)| = 1$ , we can use standard elliptic estimates to conclude that  $w_i \to w$ , in  $C^2_{loc}(\mathbb{R}^n_+)$ , for some function w, choosing a subsequence if necessary. From the identities (4.17), (4.19), (4.20) and (4.21), we see that w satisfies

$$\begin{cases} \Delta w = 0, & \text{in } \mathbb{R}^n_+, \\ \partial_n w + nU^{\frac{2}{n-2}}w = 0, & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

$$(4.23)$$

Claim.  $w(y) = O((1 + |y|)^{-1})$ , for  $y \in \mathbb{R}^{n}_{+}$ .

Choosing  $\delta > 0$  sufficiently small, we can consider the Green's function  $G_i$  for the conformal Laplacian  $L_{\hat{g}_i}$  in  $B^+_{\delta\epsilon_i^{-1}}$  subject to the boundary conditions  $B_{\hat{g}_i}G_i = 0$ on  $\partial' B^+_{\delta\epsilon_i^{-1}}$  and  $G_i = 0$  on  $\partial^+ B^+_{\delta\epsilon_i^{-1}}$ . Let  $\eta_i$  be the inward unit normal vector to  $\partial^+ B^+_{\delta\epsilon_i^{-1}}$ . Then the Green's formula gives

$$w_{i}(y) = -\int_{B_{\delta\epsilon_{i}^{-1}}^{+}} G_{i}(\xi, y) Q_{i}(\xi) \, dv_{\hat{g}_{i}}(\xi) + \int_{\partial^{+}B_{\delta\epsilon_{i}^{-1}}^{+}} \frac{\partial G_{i}}{\partial \eta_{i}}(\xi, y) w_{i}(\xi) \, d\sigma_{\hat{g}_{i}}(\xi) + \int_{\partial^{\prime}B_{\delta\epsilon_{i}^{-1}}^{+}} G_{i}(\xi, y) \left( b_{i}(\xi) w_{i}(\xi) - \overline{Q}_{i}(\xi) \right) \, d\sigma_{\hat{g}_{i}}(\xi) \,.$$
(4.24)

Using the estimates (4.19), (4.20) and (4.22) in the equation (4.24), we obtain

$$\begin{split} |w_{i}(y)| &\leq C\Lambda_{i}^{-1}\epsilon_{i}^{2}\int_{B_{\delta\epsilon_{i}^{-1}}^{+}}|\xi-y|^{2-n}(1+|\xi|)^{2-n}d\xi \\ &+ C\int_{\partial' B_{\delta\epsilon_{i}^{-1}}^{+}}|\bar{\xi}-y|^{2-n}(1+|\bar{\xi}|)^{-2}d\bar{\xi} + C\Lambda_{i}^{-1}\epsilon_{i}^{2}\int_{\partial' B_{\delta\epsilon_{i}^{-1}}^{+}}|\bar{\xi}-y|^{2-n}(1+|\bar{\xi}|)^{2-n}d\bar{\xi} \\ &+ C\Lambda_{i}^{-1}\tau_{i}\int_{\partial' B_{\delta\epsilon_{i}^{-1}}^{+}}|\bar{\xi}-y|^{2-n}(1+|\bar{\xi}|)^{1-n}d\bar{\xi} + C\Lambda_{i}^{-1}\epsilon_{i}^{n-2}\int_{\partial^{+}B_{\delta\epsilon_{i}^{-1}}^{+}}|\xi-y|^{1-n}d\sigma(\xi)\,, \end{split}$$

for  $|y| \leq \delta \epsilon_i^{-1}/2$ . Here, we have used the fact that  $|G_i(x,y)| \leq C |x-y|^{2-n}$  for  $|y| \leq \delta \epsilon_i^{-1}/2$  and, since  $v_i(y) \leq CU(y)$ ,  $|w_i(y)| \leq C\Lambda_i^{-1}\epsilon_i^{n-2}$  for  $|y| = \delta \epsilon_i^{-1}$ . Hence,

$$|w(y)| \le C\Lambda_i^{-1}\epsilon_i^2(\delta\epsilon_i^{-1})^{4-n} + C(1+|y|)^{-1} + C\Lambda_i^{-1}\epsilon_i^2\log(\delta\epsilon_i^{-1}) + C\Lambda_i^{-1}\tau_i(1+|y|)^{2-n} + C\Lambda_i^{-1}\epsilon_i^{n-2},$$

which gives

$$|w_i(y)| \le C \left( (1+|y|)^{-1} + \Lambda_i^{-1} \epsilon_i + \Lambda_i^{-1} \tau_i \right)$$
(4.25)

for  $|y| \leq \delta \epsilon_i^{-1}/2$ . The Claim now follows from the hypothesis (4.17).

Now, we can use the claim above and Lemma 4.2.1 to see that

$$w(y) = \sum_{j=1}^{n-1} c_j \partial_j U(y) + c_n \left( \frac{n-2}{2} U(y) + y^b \partial_b U(y) \right) ,$$

for some constants  $c_1, ..., c_n$ . It follows from the identity (4.15) that  $w_i(0) = \frac{\partial w_i}{\partial y_j}(0) = 0$  for j = 1, ..., n - 1. Thus we conclude that  $c_1 = ... = c_n = 0$ . Hence,  $w \equiv 0$ . Since  $|w_i(y_i)| = 1$ , we have  $|y_i| \to \infty$ . This, together with the hypothesis (4.17), contradicts the estimate (4.25), since  $|y_i| \le \delta \epsilon_i^{-1}/2$ , and concludes the proof of Lemma 4.5.1.  $\Box$ 

**Lemma 4.5.2.** There exists C > 0 such that  $\tau_i \leq C\epsilon_i$ .

*Proof.* Suppose, by contradiction, the result is false. Then we can suppose that  $\tau_i^{-1}\epsilon_i \to 0$  and, by Lemma 4.5.1, there exists C > 0 such that

$$|v_i - U - \phi_i|(y) \le C\tau_i$$
, for  $|y| \le \delta\epsilon_i^{-1}$ 

Define

$$w_i(y) = \tau_i^{-1}(v_i - U - \phi_i)(y), \quad \text{for } |y| \le \delta \epsilon_i^{-1}$$

Then  $w_i$  satisfies the equations (4.18) with

$$\begin{split} b_i &= (n-2)\hat{f}_i^{-\tau_i} \frac{v_i^{p_i} - (U+\phi_i)^{p_i}}{v_i - (U+\phi_i)},\\ Q_i &= -\tau_i^{-1} \left\{ (L_{\hat{g}_i} - \Delta)(U+\phi_i) + \Delta\phi_i \right\},\\ \overline{Q}_i &= -\tau_i^{-1} \left\{ (n-2)\hat{f}_i^{-\tau_i}(U+\phi_i)^{p_i} - (n-2)U^{\frac{n}{n-2}} - nU^{\frac{2}{n-2}}\phi_i - \frac{n-2}{2}h_{\hat{g}_i}(U+\phi_i) \right\}. \end{split}$$

Similarly to the estimates (4.19) and (4.20) we have

$$|Q_i(y)| \le C\tau_i^{-1}\epsilon_i^2 (1+|y|)^{2-n}, \qquad (4.26)$$

$$|\overline{Q}_i(y)| \le C\tau_i^{-1}\epsilon_i^2(1+|y|)^{2-n} + C(1+|y|)^{1-n}$$
(4.27)

and  $b_i$  satisfies the estimate (4.22).

By definition,  $w_i \leq C$  and, by elliptic standard estimates, we can suppose that  $w_i \to w$  in  $C^2_{loc}(\mathbb{R}^n_+)$  for some function w. By the identity (4.21) and the estimates (4.26) and (4.27) we see that w satisfies the equations (4.23).

A contradiction is achieved following the same lines as [2, Lemma 6.2].

**Proposition 4.5.3.** There exist  $C, \delta > 0$  such that

$$|\nabla^k (v_i - U - \phi_i)(y)| \le C\epsilon_i (1 + |y|)^{-k}$$

for all  $|y| \leq \delta \epsilon_i^{-1}$  and k = 0, 1, 2.

*Proof.* The estimate with k = 0 follows from Lemmas 4.5.1 and 4.5.2. The estimates with k = 1, 2 follow from elliptic theory.

### 4.6 The Pohozaev sign restriction

In this section we assume n = 3 and prove a sign restriction for an integral term in Proposition 4.3.1 and some consequences for the blow-up set. **Theorem 4.6.1.** Let  $x_i \to x_0$  be an isolated simple blow-up point for the sequence  $\{u_i \in \mathcal{M}_i\}$ . Suppose that  $u_i(x_i)u_i \to G$  away from  $x_0$ , for some function G. Then

$$\liminf_{r \to 0} P'(G, r) \ge 0.$$
(4.28)

*Proof.* Set  $v_i(y) = \epsilon_i^{\frac{1}{p_i-1}} u_i(\epsilon_i y)$  for  $y \in B^+_{\delta \epsilon_i^{-1}} = B^+_{\delta \epsilon_i^{-1}}(0)$ . We know that  $v_i$  satisfies

$$\begin{cases} L_{\hat{g}_i} v_i = 0, & \text{in } B_{\delta \epsilon_i^{-1}}^+, \\ B_{\hat{g}_i} v_i + (n-2) \hat{f}_i^{-\tau_i} v_i^{p_i} = 0, & \text{on } \partial' B_{\delta \epsilon_i^{-1}}^+, \end{cases}$$

where  $\hat{f}_i(y) = f_i(\epsilon_i y)$  and  $\hat{g}_i$  is the metric with coefficients  $(\hat{g}_i)_{kl}(y) = (g_i)_{kl}(\epsilon_i y)$ . Observe that, from Remark 4.4.11, we know that  $v_i \leq CU$  in  $B^+_{\delta \epsilon_i^{-1}}$ .

We write the Pohozaev identity of Proposition 4.3.1 as

$$P(u_i, r) = F_i(u_i, r) + \bar{F}_i(u_i, r) + \frac{\tau_i}{p_i + 1} Q_i(u_i, r), \qquad (4.29)$$

where

$$F_{i}(u,r) = -\int_{B_{r}^{+}} (z^{b}\partial_{b}u + \frac{n-2}{2}u)(L_{g_{i}} - \Delta)u\,dz,$$
  

$$\bar{F}_{i}(u,r) = \frac{n-2}{2}\int_{\partial'B_{r}^{+}} (\bar{z}^{b}\partial_{b}u + \frac{n-2}{2}u)h_{g_{i}}u\,d\bar{z},$$
  

$$Q_{i}(u,r) = \frac{(n-2)^{2}}{2}\int_{\partial'B_{r}^{+}} f_{i}^{-\tau_{i}}u^{p_{i}+1}d\bar{z} - (n-2)\int_{\partial'B_{r}^{+}} (\bar{z}^{k}\partial_{k}f)f_{i}^{-\tau_{i}-1}u^{p_{i}+1}d\bar{z}.$$

Since we can assume h(0) = 0, we have

$$\bar{F}_i(u_i, r) = O(\epsilon^{n-2}r).$$

On the other hand, we can choose r > 0 small such that  $Q_i(u_i, r) \ge 0$ . So we only have to handle  $F_i(u_i, r)$ .

Set 
$$\check{U}_i(z) = \epsilon_i^{-\frac{1}{p_i-1}} U(\epsilon_i^{-1}z)$$
 and  $\check{\phi}_i(z) = \epsilon_i^{-\frac{1}{p_i-1}} \phi_i(\epsilon_i^{-1}z)$ . We have  
 $F_i(u_i, r) = -\int_{B_r^+} (z^b \partial_b u_i + \frac{n-2}{2} u_i) (L_{g_i} - \Delta) u_i dz$   
 $= -\epsilon_i^{-\frac{2}{(p_i-1)}+n-2} \int_{B_{r_i}^+} (y^b \partial_b v_i + \frac{n-2}{2} v_i) (L_{\hat{g}_i} - \Delta) v_i dy$ ,

$$F_{i}(\check{U}_{i}+\check{\phi}_{i},r) = -\int_{B_{r}^{+}} (z^{b}\partial_{b}\check{U}_{i}+\frac{n-2}{2}\check{U}_{i})(L_{g_{i}}-\Delta)\check{U}_{i}dz$$
  
$$= -\epsilon_{i}^{-\frac{2}{(p_{i}-1)}+n-2}\int_{B_{r\epsilon_{i}^{-1}}^{+}} \left(y^{b}\partial_{b}(U+\phi_{i})+\frac{n-2}{2}(U+\phi_{i})\right)(L_{\hat{g}_{i}}-\Delta)(U+\phi_{i})dy.$$

Observe that  $\epsilon_i^{-\frac{2}{p_i-1}+n-2} = \epsilon_i^{-(n-2)\frac{\tau_i}{p_i-1}} \to 1$ , as  $i \to \infty$ , by Proposition 4.4.12. It follows from Proposition 4.5.3 that

$$|F_i(u_i, r) - F_i(\check{U}_i + \check{\phi}_i, r)| \le C\epsilon_i^2 \int_{B_{r\epsilon_i^{-1}}^+} (1 + |y|)^{1-n} dy \le C\epsilon_i r \,. \tag{4.30}$$

We know from (4.6) that  $g^{kl}(z) = \delta_{kl} + 2\pi_{kl}(0)z_n + O(|z|^2)$  in Fermi coordinates, and recall that we are assuming  $\operatorname{tr}(\pi_{kl}(0)) = h(0) = 0$ . Thus, due to symmetry arguments,

$$F_i(\check{U}_i + \check{\phi}_i, r) = O(\epsilon_i r)$$
.

Hence,  $P(u_i, r) \ge -C\epsilon_i r$ , which implies that

$$P'(G,r) = \lim_{i \to \infty} \epsilon_i^{-\frac{2}{p_i-1}} P(u_i,r) \ge -Cr.$$

Once we have proved Theorem 4.6.1, the next two propositions are similar to [36, Lemma 8.2, Proposition 8.3] or [42, Propositions 4.1 and 5.2].

**Proposition 4.6.2.** Let  $x_i \to x_0$  be an isolated blow-up point for the sequence  $\{u_i \in \mathcal{M}_i\}$ . Then  $x_0$  is an isolated simple blow-up point for  $\{u_i\}$ .

**Proposition 4.6.3.** Let  $\beta$ , R, u,  $C_0(\beta, R)$  and  $\{x_1, ..., x_N\} \subset \partial M$  be as in Proposition 4.4.7. If  $\beta$  is sufficiently small and R is sufficiently large, then there exists a constant  $\bar{C}(\beta, R) > 0$  such that if  $\max_{\partial M} u \geq C_0$  then

$$d_{\bar{g}}(x_j, x_k) \ge \bar{C} \quad \text{for all } 1 \le j \ne k \le N.$$

**Corollary 4.6.4.** Suppose the sequence  $\{u_i \in \mathcal{M}_i\}$  satisfies  $\max_{\partial M} u_i \to \infty$ . Then  $p_i \to n/(n-2)$  and the set of blow-up points is finite and consists only of isolated simple blow-up points.

# 4.7 Proof of Theorem 4.1.1

In view of standard elliptic estimates and Harnack inequalities, we only need to prove that  $||u||_{C^0(\partial M)}$  is bounded from above (see [33, Lemma A.1] for the boundary Harnack inequality). Assume by contradiction there exists a sequence  $u_i$  of positive solutions of (4.1) such that

$$\max_{\partial M} u_i \to \infty \quad \text{as } i \to \infty.$$

It follows from Corollary 4.6.4 that we can assume  $u_i$  has N isolated simple blow-up points

$$x_i^{(1)} \to x^{(1)}, \dots, x_i^{(N)} \to x^{(N)},$$

and that  $\tau_i = \frac{n}{n-2} - p_i \to 0$  as  $i \to \infty$ . Without loss of generality, suppose

$$u_i(x_i^{(1)}) = \min\left\{u_i(x_i^{(1)}), ..., u_i(x_i^{(N)})\right\}$$
 for all *i*.

Now for each k = 1, ..., N, consider the Green's function  $G_k$  for the conformal Laplacian  $L_g$  with boundary condition  $B_g G_k = 0$  and singularity at  $x^{(k)} \in \partial M$ . In Fermi coordinates centered at the respective singularities, those functions satisfy

$$|G_k(x) - |x|^{2-n}| \le C |\log |x||$$
 for  $n = 3$ ,

according to [6, Proposition B.2].

It follows from the upper bound (a) of Proposition 4.4.10 that there exists some function G such that  $u_i(x_i^{(1)})u_i \to G$  in  $C^2_{loc}(M \setminus \{x^{(1)}, ..., x^{(N)}\})$ . Moreover, the lower control (b) of that proposition and elliptic theory yields the existence of  $a_k > 0$ , k = 1, ..., N, and  $b \in C^2(M)$  such that

$$G = \sum_{k=1}^{N} a_k G_k + b,$$

and

$$\begin{cases} L_g b = 0, & \text{in } M, \\ B_g b = 0, & \text{on } \partial M. \end{cases}$$

The hypothesis  $Q(M, \partial M) > 0$  ensures that  $b \equiv 0$ . If  $\hat{g} = G_1^{\frac{4}{n-2}}g$ , by Proposition 4.3.5,  $(M \setminus \{x^{(1)}\}, \hat{g})$  is an asymptotically flat manifold (in the sense of Definition 4.3.3) with with mass

$$m(\hat{g}) = \lim_{\rho \to 0} I(x^{(1)}, \rho).$$

Moreover, we have  $R_{\hat{g}} = -\frac{4(n-1)}{n-2}G^{\frac{n+2}{n-2}}L_gG = 0$  and  $h_{\hat{g}} = -\frac{2}{n-2}G^{\frac{n}{n-2}}B_gG = 0$ . Then the positive mass Theorem 4.3.4 and the assumption that M is not conformally equivalent to  $B^3$  gives  $m(\hat{g}) > 0$ . So, by Proposition 4.3.6,

$$\lim_{\rho \to 0} P'(G_1, \rho) < 0.$$

This contradicts the local sign restriction of Theorem 4.6.1 and ends the proof of Theorem 4.1.1.
## CONCLUSION AND FUTURE PERSPECTIVES

In this thesis, we obtained both existence results of blowing-up solutions to Yamabe-type equations on manifolds and a compactness result to the boundary Yamabe problem in the scalar flat case. Despite the progress made in the field in recent years, there are many interesting and challenging questions which remain open on this type of problems. Here are some of them which I would like to investigate in the future.

**Problem 1.** Do we have compactness to the boundary Yamabe problem in the scalar flat case for higher dimensions  $4 \le n \le 24$ ? A key to this problem would be to find a better correction term in the blow-up estimates. This would be the extension to our work [5] in higher dimensions.

**Problem 2.** Do we have compactness to the boundary Yamabe problem with prescribed zero mean curvature on general manifolds? The case with umbilic boundary was solved by Disconzi and Khuri [19]. A key here would be to prove a conjecture on the vanishing of the umbilicity tensor up to a certain order.

**Problem 3.** Under what conditions do we have compactness or noncompactness to the Yamabe-type problems with both nonlinearities on the interior and boundary part of the equations? Han and Li [33] proved compactness for locally conformally flat manifolds with umbilic boundary in this case. It would be interesting to attempt constructing blowing-up solutions and proving compactness under suitable conditions for the general problem.

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